Supersymmetry as a probe of the topology of manifolds*

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Abstract

These lectures are a brief introduction to Topological field theories. Many technical details have been omitted with the hope of providing the reader a flavour of the results rather than the details. In the first lecture we discuss how supersymmetric quantum mechanics provides connections with the Atiyah-Singer Index theorems as well as topological invariants such as Euler classes of vector bundles. In the second lecture, we show how one can construct supersymmetric field theories whose observables are topological invariants on some moduli space. These topological field theories can be constructed using a standard procedure due to Witten called twisting of $N = 2$ supersymmetric field theories.

I have organised the two lectures to follow the historical sequence. The application of supersymmetry to probe topology has occured in two distinct phases. The first phase occurred in the early 80's starting from the work of Witten on supersymmetry breaking and Morse theory[1, 2]. Witten's work was extended by Alvarez-Gaumé[3] and independently by Friedan and Windey[4] to provide "proofs" of the Atiyah-Singer Index theorem for various elliptic complexes. The second phase occurred in the late 80's and early 90's and is still not over.

Unlike the first phase, where supersymmetry provided a different way of understanding well known mathematical results, the second phase has led to new results which were not known earlier to mathematicians. It has led to this brand of physics to be labelled as *experimental Mathematics* by some mathematicians. It is experimental because the methodology employed is not standard (as yet!) in Mathematics and needs to be substantiated by "proofs" in the conventional sense. Of course, it is possible that the use of quantum field theory in mathematics might become legitimate in the future. Some of the major advances in the second phase has been the quantum field theoretic understanding of Jones' polynomial invariants for knots[5] and Donaldson's invariants for four manifolds [6]. The quantum field theory approach provided rich generalisations of Jones' invariants right away. However, this was not the case with Donaldson invariants until recently.

This changed towards the end of 1994 when Seiberg and Witten[7] succeeded in describing N=2 supersymmetric $SU(2)$ gauge theory in its strong coupling

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limit. Using a version of electric-magnetic duality, they were able to construct a theory whose weak coupling limit was the strong coupling limit of $SU(2)$ gauge theories. Putting it differently, supersymmetric QCD could be solved! As we shall see later, given a N=2 supersymmetric theory, we can construct a topological field theory from it. Donaldson theory is the topological field theory obtained from the weak coupling limit of N=2 supersymmetric Yang-Mills. Seiberg and Witten's result implied that we could also study the theory in its strong coupling limit too[8]. It turns out that this theory is a lot simpler because one ends up dealing with abelian gauge fields rather than something non-abelian like $SU(2)$. From the mathematical viewpoint, this implies that a lot of proofs of Donaldson, which are very long and difficult to follow (even for an accomplished mathematician) become simple enough for graduate students to understand.

In lecture one, we will discuss how to use supersymmetric quantum mechanics to obtain topological invariants using the Euler characteristic as an example. In addition, we will briefly discuss how one can prove the Atiyah-Singer Index theorem using supersymmetric quantum mechanics.

In lecture two, we will discuss cohomological topological field theories (TFT) of which Donaldson theory is the prototype. We will not be able to discuss topological field theories of the Schwarz type of which Chern-Simons theory is the prototype[5]. It is interesting to note that cohomological field theories naturally occur in even dimensions while TFT's of the Schwarz type occur in odd dimensions. (This is not true rigorously, since one can always use dimensional reduction to obtain cohomological TFT's in odd dimensions.) A TFT is defined to be a quantum field theory which is independent of the metric and whose observables are also independent of the metric. In Chern-Simons theory, the observables are Wilson loops which are gauge invariant and are independent of the metric (since they are obtained from one-forms integrated over one-cycles).

Lecture 1

1.1 Supersymmetry: Basics

Particles are distinguished by their statistics. In $d > 2$, they come in two types: Bosons and Fermions. The spin-statistics theorem relates the spin of the particle to its statistics (unitarity being assumed). Thus, in four dimensions, particles with half-integer spin are Fermions and particles with integer spin are Bosons. Supersymmetry is a symmetry which maps Fermions to Bosons and Bosons to Fermions.

$$\boxed{\text{Bosons}} \xleftrightarrow{\text{Supersymmetry}} \boxed{\text{Fermions}}$$

In its simplest form, supersymmetry can be written as

$$[Q, \phi] = \psi \quad , \tag{1.1}$$

where ϕ is a scalar field (boson) and ψ is a spinor (fermion). Lorentz invariance of eqn. (1.1) implies that Q should transform like a spinor. Further, it should be an anti-commuting variable (Grassmann). Thus, supersymmetry introduces new conserved charges which are spinorial and anti-commuting. In even dimensions, fermions have definite chirality (as given by γ^5 in 4d). Thus, the supersymmetry generators also can have definite chirality. The Lorentz group in 4d is $SO(4) \sim SU(2)_L \times SU(2)_R$. Thus finite dimensional representations of the Lorentz group can be classified by the spin quantum numbers associated with each of the $SU(2)$'s. Let us label them by (j_L, j_R). The quantum numbers associated with usual particles/fields are

Field	(j_L, j_R)	dim. of rep.
Scalar ϕ	$(0,0)$	0
Weyl Fermion ψ_α or $\psi_{\dot\alpha}$	$(\frac{1}{2},0)$ or $(0,\frac{1}{2})$	2
Dirac Fermion $(\psi_\alpha, \psi_{\dot\alpha})$	$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	4
Gauge Field A_μ	$(\frac{1}{2},\frac{1}{2})$	4
Antisymmetric Tensor $B_{\mu\nu}$	$(1,0) \oplus (0,1)$	6

Thus, we see that the supersymmetry charge comes in two types: Q_α which transforms as a $(\frac{1}{2},0)$ and $Q_{\dot\alpha}$ which transforms as a $(0,\frac{1}{2})$ under the Lorentz group. Eqn. (1.1) can be rewritten as

$$\begin{aligned} [Q_\alpha, \phi] &= \psi_\alpha \quad , \\ [Q_{\dot\alpha}, \phi] &= \psi_{\dot\alpha} \quad . \end{aligned} \tag{1.2}$$

In an abstract fashion, one can ask how the usual Poincaré algebra can be extended to include the supersymmetry generators. This leads to the following super-Lie algebra (which satisfies graded Jacobi identities since it involves both commutators and anti-commutators). This leads to the following symmetry algebra

$$\{Q^I_\alpha, \bar{Q}_{\dot\alpha J}\} = \delta^I_J \sigma^\mu_{\alpha\dot\alpha} P_\mu \quad , \tag{1.3}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 0 \quad , \tag{1.4}$$

$$[Q^I_\alpha, P_\mu] = 0 \quad , \tag{1.5}$$

where $I, J = 1, \ldots, N$ are the number of supersymmetries, $\sigma^\mu = (1, \vec\sigma)$ and $\bar{Q}_{\dot\alpha I} = (Q^I_\alpha)^\dagger$. We have left out the obvious part of the algebra and not included central charges. The interested reader is referred to [9] for more details.

In particular, we see that $\mathrm{Tr}\{Q, \bar{Q}\} \propto H$, where we have used $H = P^0$ and the trace is in "spin space". Consider a theory which is supersymmetric and we are interested in finding out whether supersymmetry is spontaneously broken or not. Unbroken supersymmetry implies that

$$Q|0\rangle = 0 \quad , \quad \bar{Q}|0\rangle = 0 \quad ,$$

where $|0\rangle$ is the ground state of the theory. Using $\mathrm{Tr}\{Q, \bar{Q}\} \propto H$, we see that unbroken supersymmetry implies that $E = 0$ for the ground state. One can also show that $E \geq 0$ for a generic state. In addition, one can see that $E = 0$ implies that

$$P^i|0\rangle = 0 \quad .$$

(Use $det[Q, \bar{Q}] \propto (H^2 - P^2) \geq 0$ for any state.) Thus for the purpose of supersymmetry breaking it is sufficient to consider states with $P^i = 0$. The supersymmetry algebra simplifies in the subspace of such states. One obtains an effective $0 + 1$ dimensional theory where the supersymmetry algebra is

$$\{Q_I, Q_J\} = \delta_{IJ} H \quad , \tag{1.6}$$

for $I, J = 1, \ldots, 4N$. Note that the above equation is obtained from the four dimensional supersymmetric algebra after a suitable redefinition of the supersymmetry generators. The number of generators in the $0 + 1$ dimensional theory reflects the fact that it has originated from a 4 dimensional theory with N supersymmetries.

We seem to have used the words fields and states in an interchangeable way. For a generic field theory, the Hilbert space is infinite dimensional. Thus to facilitate counting of states (below a certain cut-off) one puts the system in a box with periodic boundary conditions on both the bosons and fermions. The boundary condition on fermions is fixed by supersymmetry and is different from the conventional anti-periodic boundary conditions used in finite temperature field theory. In finite volume, one does not usually talk of particles since they are sort of ill-defined. We shall instead talk of states which are bosonic or fermionic. This is fixed by their eigenvalue for the operator $exp\,(2\pi J_z)$. Thus one has $exp\,(2\pi J_z)|b\rangle = |b\rangle$ and $exp\,(2\pi J_z)|f\rangle = -|f\rangle$ for bosonic and fermionic states respectively. This operator can be represented as

$$(-)^F \equiv exp\,(2\pi J_z) \quad ,$$

where $F = 0$ on bosonic states and $F = 1$ on fermionic states.

Returning to the earlier issue as to whether supersymmetry is spontaneously broken or not, one can see that states with $E > 0$ come in pairs. Restricting our attention to one supersymmetry generator $Q \equiv Q_1$ with $Q^2 = H$, given a bosonic state $|b\rangle$ with $E > 0$, one can construct a corresponding fermionic state (normalised)

$$|f\rangle \equiv \frac{1}{\sqrt{E}} Q|b\rangle \quad ,$$

where we assume that the bosonic state is normalised. It is clear that this pairing is broken when $E = 0$. Then one either has $Q|b\rangle = 0$ or $Q|f\rangle = 0$. Thus, the states with $E = 0$ are not paired by supersymmetry. Let $n_B^{E=0}$ ($n_F^{E=0}$) be the number of bosonic (fermionic) states with $E = 0$.

One can see that if the parameters in the Hamiltonian such as mass, gauge couplings, volume etc. are slowly varied, states with $E > 0$ might simultaneously attain $E = 0$ or two states with $E = 0$ might pair up and form a $E > 0$ state[1]. This implies that the difference

$$(n_B^{E=0} - n_F^{E=0}) \quad ,$$

is independent of changes in the various parameters of the theory. Thus this is an object which one can calculate in a reliable fashion. Further, if $(n_B^{E=0} - n_F^{E=0}) \neq 0$, supersymmetry is unbroken, because there exists atleast one state with $E = 0$. One can easily see that

$$\mathrm{Tr}(-)^F = (n_B^{E=0} - n_F^{E=0}) \quad , \tag{1.7}$$

where the trace is over all states in the Hilbert space. It is easy to see that states with $E > 0$ cancel out since they come in pairs of fermions and bosons. This object is called the *Witten Index*. In addition, due to the exact cancellation of the $E > 0$ states, one can replace $\mathrm{Tr}(-)^F$ by $\mathrm{Tr}(-)^F exp(-\beta H)$. It is clear that both will give the same result. However, the second version can be thought of as a regularised version of the first one. As we shall see later, the $\beta = 0$ and $\beta \neq 0$ computations give rise to different looking formulae for the same object.

1.2 Witten Index as the index of an operator

The Witten Index can be interpreted as the index of an operator. This follows from the following argument. The Hilbert space $\mathcal{H}$of a quantum field theory (after appropriately regularising) can be split into bosonic and fermion subspaces $\mathcal{H}_B$ and $\mathcal{H}_F$ respectively. These subspaces are basically the $F = 0$ and $F = 1$ subspaces.

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \quad .$$

Since the supersymmetry operator Q maps one from fermionic states to bosonic and vice-versa, it can be represented as an off-diagonal operator

$$Q = \begin{pmatrix} 0 & N^* \\ N & 0 \end{pmatrix} \quad , \tag{1.8}$$

if the states are arranged as $\begin{pmatrix} B \\ F \end{pmatrix}$. Since Q is Hermitian, the operator N^* is the complex conjugate of N as indicated in the above equation. Since $H = Q^2$, the zero energy bosonic states satisfy $N\psi = 0$ and the zero energy fermionic states satisfy $N^*\psi = 0$. Thus one has

$$\mathrm{Tr}(-)^F = \dim(\ker N) - \dim(\ker N^*) \quad . \tag{1.9}$$

[1]This assumes that no new states emerge in the theory as parameters are varied.

The RHS of the above equation is the definition of the index of the operator N. (In the above equation, ker$N$ refers to the kernel of the operator N, i.e., solutions of the equation $N\psi = 0$). Atiyah and Singer related the indices of various elliptic operators to topological invariants. Thus, by suitably choosing N (or Q) in some theory, one can obtain topological invariants. In some sense, one is "proving" the Atiyah-Singer index theorems by using supersymmetry with the topological invariants as a by-product.

For example, one can choose N to be the Dirac operator, $(-)^F$ can be identified with γ^5, then the index is related to the well known chiral anomaly which occurs in even dimensions.

One represent the Witten Index as the partition function at "inverse temperature" β for an ensemble with density matrix $\rho = (-)^F exp[-\beta H]$. Using the statistical mechanics - path integral relationship, one can write the Witten Index as

$$\mathrm{Tr}(-)^F e^{-\beta H} = \int_{PBC} [d\phi d\psi] e^{-S_E(\phi,\psi)} \quad , \tag{1.10}$$

where PBC refers to periodic boundary conditions on both fermionic and bosonic fields and ϕ and ψ represent bosonic and fermionic fields respectively.

1.3 Witten Index as the Euler characteristic of a manifold

The Euler characteristic for a manifold M can be defined in many ways. It is one of the simplest topological invariants one can define for a manifold. For simplicity, consider a two dimensional manifold. The Euler characteristic can be defined by counting the number of vertices n_v, number of edges n_e and number of faces n_f and using the following formula

$$\chi(M) = n_v - n_e + n_f \quad , \tag{1.11}$$

which is clearly something which is topological (does not need the metric). The above expression can be rewritten as $\chi(M) = \sum(-)^p b_p$, where b_p are the Betti numbers of M. However, there is another definition of the Euler characteristic which is not patently topological.

$$\chi(M) = -\frac{1}{8\pi}\int_M R \tag{1.12}$$

where R is the Ricci scalar. The Ricci scalar is constructed from the Riemann tensor which is obtained from the metric. Thus these two formulae seem quite different. However, as we will see that the two versions correspond to evaluating the Witten index with $\beta = 0$ and $\beta \neq 0$. Since the Witten index is independent of β as we argued earlier, one can see that the two definitions are in fact equivalent.

We shall now illustrate the above with a concrete example. This example has two supersymmetry generators. Consider a n-dimensional Riemannian manifold

M with metric g_{ij}. Let $\phi^i(t)$ be maps of R or S^1 to M. The bosonic σ- model is given by

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j \quad , \quad \text{where } \dot{\phi} = \partial_t \phi \text{ and } i = 1, \ldots, n \quad . \tag{1.13}$$

The supersymmetric generalisation is

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}(\phi)\bar{\Psi}^i\gamma^0 \frac{D\Psi^j}{dt} + \frac{1}{12} R_{ijkl}\bar{\Psi}^i\Psi^k\bar{\Psi}^j\Psi^l \quad , \tag{1.14}$$

where R_{ijkl} is the Riemann tensor and $\frac{D}{dt}\Psi^i = \frac{d}{dt}\Psi^i + \Gamma^i_{jk}\dot{\phi}^i\Psi^k$ and $\bar{\Psi}^i_\alpha = \bar{\Psi}^i_\beta \gamma^0_{\beta\alpha}$ with $\alpha, \beta = 1, 2$. Ψ^i is a two component real spinor with $\gamma^0 = \sigma_2$. The action (1.14) is invariant under the supersymmetry transformations parametrised by ε^α.

$$\begin{aligned} \delta\phi^i &= \bar{\varepsilon}\Psi^i \\ \delta\Psi^i &= -i\gamma^0\dot{\phi}^i\varepsilon - \Gamma^i{}_{jk}\bar{\varepsilon}\Psi^j\Psi^k \quad . \end{aligned} \tag{1.15}$$

It is easy to see that the above reduces to the usual flat space supersymmetry transformations by choosing $M = R^4$ and setting the Christoffel connection to zero in the above equation.

The standard Noether construction can now be used to derive the supersymmetry charges associated with the above supersymmetry transformations. We thus obtain the following (two) supersymmetry charges (where we have represented the two component real fermion as $\Psi^i = \begin{pmatrix} \psi^i \\ \psi^{*i} \end{pmatrix}$ in a complex basis)

$$Q \sim i\Psi^{*i}P_i \quad , \tag{1.16}$$

$$Q^* \sim i\Psi^i P_i \quad , \tag{1.17}$$

where P_i are the momenta conjugate to ϕ^i. ($P_i = \frac{\delta L}{\delta\dot{\phi}^i}$) We now canonically quantize the theory. The Poisson brackets are

$$\begin{aligned} [\phi^i, P_j] &= i\delta^i_j \quad , \\ \{\psi^i, \psi^{*j}\} &= g^{ij}(\phi) \quad , \\ \{\psi^i, \psi^j\} = & \; 0 \; = \{\psi^{*i}, \psi^{*j}\} \end{aligned} \tag{1.18}$$

One can verify (using the above commutation relations) that the supersymmetry charges satisfy the following commutation relations

$$\{Q, Q\} = 0 = \{Q^*, Q^*\} \quad , \quad \{Q, Q^*\} = \text{Laplacian on } M \quad , \tag{1.19}$$

where we have identified P_i with $\frac{\delta}{\delta\phi^i}$ and thus $g^{ij}P_iP_j$ is the Laplacian on M. Comparing with the supersymmetry algebra (in $0+1$ dimensions), we see that the Hamiltonian for this model is the Laplacian on M. The Hilbert space of this theory can be constructed as follows. The ψ can be treated as fermion annihilation operators and the ψ^* as the creation operators. (In writing the Hilbert space, we work in a mixed representation, where we represent the fermionic part in the Fock representation while we leave the bosonic part in terms of the fields.) The Hilbert space can be identified with the following objects

States with no fermions	$\longleftrightarrow$	functions on M
$f(\phi)\|0\rangle$		$f(\phi)$
States with one fermion	$\longleftrightarrow$	one-forms on M
$f_i(\phi)\psi^{*i}\|0\rangle$		$f_i(\phi)d\phi^i$
States with two fermions	$\longleftrightarrow$	two-forms on M
$f_{ij}(\phi)\psi^{*i}\psi^{*j}\|0\rangle$		$f_{ij}(\phi)d\phi^i d\phi^j$
	$\vdots$	

The above table shows that the Hilbert space of this theory is in 1-1 correspondence with the exterior algebra $\Lambda^*(M)$, which is the space of forms on M. What is the action of Q on $\Lambda^*(M)$? Using the representation of Q given in (1.17) and the canonical commutation relations, one can verify the following correspondences

$$\begin{aligned} \mathcal{H} &\longleftrightarrow \Lambda^*(M) \\ Q &\longleftrightarrow d \\ Q^* &\longleftrightarrow \delta \end{aligned} \tag{1.20}$$

The Hamiltonian $H = d\delta + \delta d$ and hence $E = 0$ states are harmonic forms on M. The Betti numbers b_p give the number of p-forms which are harmonic. Further, in our model, the p-forms with even p are bosonic and odd p are fermionic states. Thus, it is easy to see that the Witten index

$$\mathrm{Tr}(-)^F = \sum(-)^p b_p = \chi(M) \quad ,$$

which gives the Euler characteristic as promised. We can also evaluate the Witten index for non-zero β using the path integral. Rather than go into the details, we will list some of the crucial steps. The interested reader can fill in the details. In our model one has

$$\mathrm{Tr}(-)^F e^{-\beta H} = \int [d\phi d\Psi] e^{-L} \quad , \tag{1.21}$$

where L is as given in (1.14). As we shall see, evaluating the path integral gives us an alternate representation for the Euler characteristic.

(i) Expand the fields $\phi(t)$ and $\psi(t)$ as a Fourier basis with respect to the time variable. The frequencies are $\omega_n = 2\pi n/\beta$.

(ii) Choosing β to be small enough, there is a large gap between the zero-mode states and the other modes (with $n > 0$).

(iii) Do the path-integral over the non-zero modes. Perturbation theory will suffice here due to the large gap.

(iv) Do the zero-mode integration exactly.

In our example, in step (iii) the fermionic determinant exactly cancels the bosonic determinant. This is due to supersymmetry and can be proved using supersymmetric Ward identities. In doing step (iv), one obtains the following integration

$$\frac{1}{(2\pi)^{d/2}}\int_M d^n\phi \prod_{m=1}^{n} d\psi^{*m} d\psi^m e^{-\frac{1}{4}R\psi^*\psi^*\psi\psi} \quad ,$$

which one expanding gives

$$\prod_{m=1}^{n} d\psi^{*m} d\psi^m [1 - \frac{1}{4}R\psi^*\psi^*\psi\psi + \frac{1}{2}(\frac{1}{4}R\psi^*\psi^*\psi\psi)^2 + \cdots] \quad . \tag{1.22}$$

In doing the fermionic integration, only the terms which "soak up" the fermionic zero-mode integrations need to be considered. (The rules of integration for a Grassmann variable ψ are $\int d\psi = 0$ and $\int d\psi\psi = 1$. Thus $\int d\psi f(\psi) = f'(0)$.) So, we obtain the following

$$\begin{aligned} \chi(M) &= 0 \text{ for odd } n \quad , \\ \chi(M) &= -\frac{1}{8\pi}\int_M R \quad \text{for } n = 2, \\ \chi(M) &= \frac{1}{32\pi^2}\int_M R \wedge R \quad \text{for } n = 4. \end{aligned} \tag{1.23}$$

The example can be modified in many ways to obtain other topological objects. For example, there is a discrete symmetry in the Lagrangian corresponding to

$$Q_5 : \quad \psi \leftrightarrow \psi^* \quad . \tag{1.24}$$

One can show that the object $\mathrm{Tr} Q_5 e^{-\beta H}$ also depends on the zero energy states. Q_5 acts on $\Lambda^*(M)$ by exchanging p-forms with $(n-p)$-forms. Thus, it corresponds to Poincaré duality. The index corresponding to this is called the Hirzebruch signature of the manifold M.

In order to obtain the index of the Dirac operator, impose the condition $\psi_1^i = \psi_2^i = \psi^i/\sqrt{2}$ in the Lagrangian (1.14). The Lagrangian then reduces to

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}(\phi)\bar{\Psi}^i\gamma^0\frac{D\Psi^j}{dt} \tag{1.25}$$

The number of supersymmetries also reduces from two to one. The generator of this supersymmetry is $Q = i\psi^i P_i$ with $Q^2 = H$. The canonical commutation relations imply that we can make the following identifications

$$\psi^i \leftrightarrow \gamma^i \quad , \quad iP_i \leftrightarrow \partial_i \quad ,$$

from which it follows that Q can be identified with the Dirac operator in the target manifold M. $(-)^F$ can be identified with γ^5. Thus the Witten index in this case can be identified with the index of the Dirac operator. We refer the reader to [4] for more details on the computation.

Thus, in this lecture, we have demonstrated that in supersymmetric quantum mechanical models, the Witten index provides us with a topological invariant. The choice of model determines the invariant. It would however, be nicer if we could create a theory where all observables are topological, thus providing a richer set of invariants. This will be the content of the next lecture.

One of the topics we have not discussed here is Morse theory which can be understood using supersymmetry. We refer the reader here to Witten's work[2] and also the some beautiful lectures on Morse theory by Atiyah and Bott[10].

Lecture 2

We summarise some of the salient features of an example that was considered in the first lecture.

- The problem was essentially in $0+1$ dimensions (supersymmetric quantum mechanics).
- Path integrals are infinite dimensional integrals in the conventional mathematical sense. However, for the cases we considered, the calculation reduced to a finite dimensional integral thus enabling us to make contact with the standard definitions of topological invariants.
- The zero modes of the fermions ψ^i behaved like an element of T^*M, the cotangent bundle of M.

  $$\psi^i \leftrightarrow d\phi^i \quad ,$$

  with the fermion number being the same as form number.
- The topological invariant we obtained was the Euler characteristic of M which is the Euler class of the vector bundle T^*M.
- The finite dimensional integral was over "T^*M"

  $$\int d\phi^i \int d\psi^i d\psi^{*i}$$

There exists a field theoretic generalisation of the above and such theories are called *Topological Quantum Field Theories*[6]. Most of the above features occur in these theories. Typically, the analog of T^*M is the moduli space (of solutions) of a set of equations. However, we will see that these theories possess a large class of observables whose correlation functions are topological invariants and hence provide a richer set of invariants than the above example. In addition, the cohomology problem becomes an *equivariant* one which means that the analog of the supersymmetry charge satisfies $Q^2 = 0$ only upto overall gauge transformation.

We now provide examples of such theories:

	Topological Field Theory	Analog of T^*M
1.	Donaldson Theory (4D)	Moduli space of self-dual Yang-Mills.
2.	Donaldson-Seiberg-Witten theory (4D)	Moduli space of "monopole" equations
3.	Topological Gravity (2D)	Moduli space of Riemann surfaces
4.	Topological Sigma Models on a Riemann surface Σ	Space of Holomorphic Maps: $\text{Map}(\Sigma, M)$
5.	Topological Yang-Mills (2D)	Space of flat connections or Moduli space of stable vector bundles

2.1 A digression: BRST invariance in gauge theories

In the example considered in lecture 1, we had

$$[Q, \phi^i] = \psi^i \quad .$$

The operator Q is a scalar operator on the "target" manifold M even though it is an anti-commuting operator. Does this violate the spin-statistics theorem? Naively, a theory with such operators is non-unitary and hence the spin-statistics theorem is not violated. Does this sort of an operator occur elsewhere? The BRST quantisation of gauge theories provides such an example. We will now briefly discuss this.

In the path integral quantisation of gauge theories, one typically picks a gauge (say, $D_\mu A^a_\mu = 0$). Then, one introduces Faddeev-Popov ghosts $c^a(x)$ which are Lie-Algebra valued anti-commuting scalars [2] and the corresponding anti-ghost $b_a(x)$. The Faddeev-Popov determinant is obtained after integrating out the ghosts. It has been discovered that the the gauge fixed action has a fermionic symmetry (called BRST after Becchi, Rouet, Stora and Tyutin) generated by a scalar charge Q given by

$$Q = \int \left(c^a J_a(x) + \frac{1}{2} f_{ab}{}^c c^a c^b b_c \right) \quad , \tag{2.1}$$

where $\int J_a(x)$ is the charge which generates residual gauge transformations (which preserve the gauge condition).

The BRST tranformation of the fields are

$$\begin{aligned} [Q, A^a_\mu] &= -D_\mu c^a \quad , \\ \{Q, c^a\} &= \frac{1}{2}[c, c] \quad . \end{aligned} \tag{2.2}$$

[2]This is assigned ghost charge $U = +1$ with the gauge field having ghost number (charge) 0.

One can check that

$$Q^2 = 0 \quad . \tag{2.3}$$

Physical states are given by[3]

$$Q|\psi\rangle = 0 \quad \text{and} \quad |\psi\rangle \neq Q|\xi\rangle \quad . \tag{2.4}$$

This is the analog of the Gauss law condition imposed in QED (in the $A_0 = 0$ gauge). Ghosts have wrong statistics and the theory which includes the complete Hilbert space is non-unitary. However, condition (2.4) is powerful enough to show that the ghosts decouple from physical processes and hence the truncated theory is unitary. Further, operators of the form $\{Q, \xi\}$ decouple from all correlation functions. Thus one has the following correspondence.

GAUGE INVARIANT OBSERVABLES	$\longleftrightarrow$	COHOMOLOGY OF THE OPERATOR Q

Thus, operators such as Q occur quite naturally in gauge theories. As is clear, the occurance of such an operator is not sufficient to obtain a topological theory.

2.2 Ingredients for a topological field theory?

The basic ingredients involved in constructing a topological theory are

(i) The theory has an operator Q such that $Q^2 = 0$.

(ii) The energy-momentum tensor is Q-exact.

$$\frac{\delta S}{\delta g_{\mu\nu}} \equiv T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\} \tag{2.5}$$

(iii) The physical observables $\mathcal{O}_i$ are in the cohomology of Q and their variation with respect to the variation of the metric is exact.

$$\delta_g \mathcal{O}_i = \{Q, \rho^i_{\mu\nu}\} \quad .$$

Assuming that the above conditions are satisfied, one can show that such a theory is topological. Consider the correlation function of n such observables $\langle \Pi_{i=1}^n \mathcal{O}_i \rangle$. The variation of such a correlation function with respect to that of the metric is given by

$$\begin{aligned} \delta_g \langle \Pi_{i=1}^n \mathcal{O}_i \rangle &= \delta_g \left[\int [d\phi] e^{-S[\phi,g]} \mathcal{O}_1 \cdots \mathcal{O}_n \right] && (2.6) \\ &= \sum_{i=1}^n \int [d\phi] e^{-S[\phi,g]} \mathcal{O}_1 \cdots \delta_g \mathcal{O}_i \cdots \mathcal{O}_n \\ &+ \int [d\phi] e^{-S[\phi,g]} \{Q, \Lambda_{\mu\nu}\} \mathcal{O}_1 \cdots \mathcal{O}_n && (2.7) \\ &= 0 && (2.8) \end{aligned}$$

[3]The corresponding condition for operators is $\{Q, \psi\} = 0$ and $\psi \neq \{Q, \xi\}$.

where in eqn. (2.7), we have assumed that the measure $[d\phi]$ is independent of the metric. Eqn. (2.8) follows from the decoupling of Q-exact states (which is always assumed though one needs to prove decoupling as in the case of gauge theories). Thus, we obtain that all correlation functions (involving operators of the type described above) are independent of the metric and hence, give rise to topological invariants. A trivial consequence of this is that the result is independent of the positions of the operators. Thus all that remains to be done is to construct a theory which satisfies the three conditions we have described and then the theory is naturally a topological field theory.

2.3 The Recipe: A Witten Twist

Witten provided a general procedure for constructing such a theory. Again, supersymmetry will prove to be the underlying reason for the theory being topological! One starts from a $N = 2$ theory and "twists" the theory to obtain a topological theory. We shall see that conditions (i) and (ii) described earlier are automatically satisfied. Typically, (iii) is trivially satisfied.

In order to understand twisting, we shall consider the $N = 2$ supersymmetry algebra in 4 dimensions[4]

$$\{Q^I_\alpha, Q^J_{\dot\alpha}\} = \delta^{IJ}\sigma^\mu_{\alpha\dot\alpha}P_\mu \quad , \quad I,J=1,2 \tag{2.9}$$

$$\{Q^I_\alpha, Q_\beta\}^J = 0 = \{Q^I_{\dot\alpha}, Q_{\dot\beta}\} \tag{2.10}$$

The algebra has a global $U(2) = SU(2)_I \times U(1)_U$ symmetry. This symmetry mixes the various supersymmetry generators and commutes with the generators of the usual Poincaré algebra (translations and Lorentz rotations). Explicitly, we have

$$[Q^I_\alpha, B_r] = (b_r)^I{}_J Q^J_\alpha \quad . \tag{2.11}$$

This symmetry is referred to as $R-invariance$. For $N = 2$ it is easy to see that $U(2)$ is the largest possible group with a non-trivial action on the supersymmetry generators. The Lorentz and $SU(2)_I$ charges (representations) of the $\{Q^I_\alpha,\, Q^J_{\dot\alpha}\}$ are

	(j_L, j_R, j_I)
Q^I_α:	$(\frac{1}{2}, 0, \frac{1}{2})$
$Q^I_{\dot\alpha}$:	$(0, \frac{1}{2}, \frac{1}{2})$

Suppose, the Lorentz $SU(2)_R$ is replaced by $SU(2)'_R$ which is the diagonal sum of $SU(2)_R$ and $SU(2)_I$. The new Lorentz group is $SU(2)_L \times SU(2)'_R$. The new choice of Lorentz group leads to unusual charges for all the supersymmetry generators. We obtain that the new charges are

[4]The procedure works in other even dimensions as well.

Old Label	(j_L, j'_R)	New Label
Q^I_α	$(\frac{1}{2}, \frac{1}{2})$	Q_μ or $Q_{\alpha\dot{\alpha}}$
$Q^I_{\dot{\alpha}i}$	$(0,0) \oplus (0,1)$	$Q \oplus Q_{[\mu\nu]}$

Thus all the half-integer spins have been converted to integer spins. Importantly, one of the supersymmetry generators transforms as a scalar i.e., with Lorentz charge $(0,0)$ which we have labelled as Q. A part of the supersymmetry algebra now becomes

$$\text{(i)}\ \{Q, Q\} = 0 \quad , \tag{2.12}$$

$$\text{(ii)}\ \{Q, Q_\mu\} \propto P_\mu \quad . \tag{2.13}$$

Thus condition (i) is satisfied by Q. We know that P_μ is part of the stress-tensor and hence eqn. (2.13) coupled with general covariance suggests that Q_μ must occur as a part of a larger object $\Lambda_{\mu\nu}$. Thus condition (ii) is also satisfied. This suggests that such a theory will be topological provided we restrict ourselves to the subset of observables which are in the cohomology of Q.[5] We will now move on to a specific example. The example we will consider is Donaldson theory which arises as a twisting of $N = 2$ topological Yang-Mills in 4 dimensions.

2.4 An example: Donaldson Theory

Donaldson studied a class of topological invariants (named after him) associated with four dimensional manifolds. Witten constructed a topological field theory by twisting $N = 2$ supersymmetric Yang-Mills whose invariants are the same as Donaldson's[6, 11]. One of his motivations was to obtain a relativistic generalisation of Floer theory (which is related to 3 manifolds but makes extensive use of instantons in 4d).

Let us consider the field/matter content of $N = 2$ supersymmetric Yang-Mills. Before twisting, the fields are the gauge field, two complex Weyl Fermions and a complex scalar field. They all transform in the adjoint representation of the gauge group G. (It is useful to assume that G=$SU(2)$.) In the table given below, U refers to the charge of the field under $U(1)_U$.

Field	Rep. before twisting $(j_L, j_R)^U$	Rep. after twisting $(j_L, j'_R)^U$
A_μ	$(\frac{1}{2}, \frac{1}{2})^0$	$(\frac{1}{2}, \frac{1}{2})^0$
$\psi_\mu, \eta, \chi_{[\mu\nu]}$	$(\frac{1}{2}, 0)^{+1} \oplus (0, \frac{1}{2})^{-1}$	$(\frac{1}{2}, \frac{1}{2})^1 \oplus (0,0)^{-1} \oplus (0,1)^{-1}$
ϕ, ϕ^*	$(0,0)^{\pm 2}$	$(0,0)^{\pm 2}$

[5] An alternative way of viewing this is to say that there is a topological sub-sector in any $N = 2$ theory.

One can see that the bosonic fields were untouched by the twisting (this is because they are invariant under $SU(2)_I$). The fermionic part consists of a vector ψ_μ, a scalar η and a anti-self-dual 2-form field $\chi_{\mu\nu}$. The scalar supersymmetry charge Q carries charge $U = 1$. This corresponds to the *ghost number*. This is the analog of fermion number in the supersymmetric QM example. The tranformation of the fields under the scalar supersymmetry can be obtained from the usual $N = 2$ transformations. They are

$$
\begin{aligned}
[Q, A^a_\mu] &= \psi^a_\mu && (2.14)\\
\{Q, \psi^a_\mu] &= -(D_\mu\phi)^a && (2.15)\\
[Q, \phi^a] &= 0 && (2.16)\\
[Q, \lambda] &= 2i\eta && (2.17)\\
\{Q, \eta\} &= \frac{1}{2}[\phi, \lambda] && (2.18)\\
\{Q, \chi_{\mu\nu}\} &= (F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}) && (2.19)
\end{aligned}
$$

Note that one has to set $\lambda = \phi^*$ in the above equations to match with $N = 2$ Yang-Mills. One can see that $[Q^2, A^a_\mu] = -(D_\mu\phi)^a$. Thus $Q^2 \neq 0$. However, $D_\mu\phi$ is a (field-dependent) gauge transformation. Thus we have

$$Q^2 = 0 \quad \text{upto gauge transformations.}$$

This is unlike the BRST charge for gauge transformations. The cohomology of such a charge is referred to as *Equivariant Cohomology*.
Note: The supersymmetry algebra we have discussed is on R^4. However, generically curved manifolds do not admit spinors (the obstruction being the second Stiefel-Whitney class). The twisted version however does not see any such topological obstruction because all the spinors now transform as bosons. In the Lagrangian which we will study later, one can explicitly verify invariance under the scalar supersymmetry transformation without ever requiring the vanishing of the Riemann tensor in any of the variations (as is the case for untwisted supersymmetry).

The Lagrangian for the supersymmetric Yang-Mills theory is given by

$$
\begin{aligned}
\mathcal{L}_0 = \int_M d^4x\sqrt{g}Tr\Big\{&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\phi D_\mu D^\mu\lambda - i\eta D_\mu\psi^\mu + iD_\mu\psi_\nu\chi^{\mu\nu}\\
&-\frac{i}{8}\phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\} - \frac{i}{2}\phi^*\{\psi_\mu, \psi^\mu\} - \frac{i}{2}\{\eta, \eta\} - \frac{1}{8}[\phi, \phi^*]^2\Big\} \qquad (2.20)
\end{aligned}
$$

One can add a term of the form $\int F\wedge F$ to the action. This term is a purely topological term and gives the Pontryagin index. Thus the theory remains topological. The Lagrangian for Donaldson theory is

$$\mathcal{L} = \frac{1}{e^2}\left[\mathcal{L}_0 + \frac{1}{4}\int_M TrF\wedge F\right] \quad , \qquad (2.21)$$

where we have introduced a coupling constant e into the Lagrangian. One can check that the Lagrangian in (2.21) is Q-exact, i.e., it can be written as $\frac{1}{e^2}\{Q, V\}$ where $V = \frac{1}{4}TrF_{\mu\nu}\xi^{\mu\nu} + \frac{1}{2}Tr\psi_\mu D^\mu\lambda - \frac{1}{4}Tr(\eta[\phi, \lambda])$. Further, as claimed, the stress-tensor $T_{\mu\nu}$ is also Q-exact.

This implies that all correlation functions are independent of the metric (following our earlier arguments) as well as on the coupling constant e. For example, consider the variation of the partition function of the theory with respect to the coupling constant,

$$\begin{aligned} \frac{\delta Z}{\delta[\frac{1}{e^2}]} &= -\int [DA\cdots]e^{-\mathcal{L}}\{Q, V\} \\ &= -\langle\{Q, V\}\rangle = 0 \end{aligned} \tag{2.22}$$

Thus the partition function is independent of the value of e. Thus we can choose to study the theory in the $e \to 0$ limit where the path integral is dominated by the minima of the action. Concentrating on the gauge part of the action,

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= \frac{1}{4e^2}\int_M \sqrt{g}Tr(F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}{}^*F^{\mu\nu}) \\ &= \frac{1}{4e^2}\int_M \sqrt{g}Tr(F_{\mu\nu} + {}^*F_{\mu\nu})(F^{\mu\nu} + {}^*F^{\mu\nu}) \quad , \end{aligned} \tag{2.23}$$

which is minimised when $F_{\mu\nu} = -{}^*F_{\mu\nu}$. Thus the minima occurs for anti-self dual instanton solutions. This is precisely what Donaldson studied in the context of his theory of four-manifold invariants. This suggests that one is on the right track in constructing a Topological Field Theory whose observables will be Donaldson's invariants.

Instanton solutions may or may not exist depending on the choice of manifold M and the choice of gauge group G. If they do exist, then the instantons have a *moduli space* $\mathcal{M}$ which is smooth except for mild singularities. For the case of $G = SU(2)$, the *formal* dimension of $\mathcal{M}$ is given by the formula

$$d(\mathcal{M}) = 8p_1(E) - \frac{3}{2}(\chi(M) + \sigma(M)) \quad , \tag{2.24}$$

where $p_1(E)$ is the first Pontryagin number of the vector bundle for G, $\chi(M)$ and $\sigma(M)$ are the Euler characteristic and Hirzebruch signature of the manifold M. The moduli space can be thought of as the space

$$\mathcal{M} = \{\text{Space of Anti} - \text{Self Dual Connections on M}\}/\mathcal{G} \quad ,$$

where $\mathcal{G}$ is the set of gauge transformations.

Suppose, one finds an instanton solution $\bar{A}$. We can expand the connection A as $\bar{A} + \delta A$ to probe for neighbourhood solutions which also satisfy the anti-self duality condition. This condition implies the following condition on δA

$$D_\mu\delta A_\nu - D_\nu\delta A_\mu + \epsilon_{\mu\nu\rho\sigma}D^\rho\delta A^\sigma = 0 \quad , \tag{2.25}$$

where the covariant derivatives D_μ are taken with respect to the background gauge field $\bar{A}$. Among the variations δA, there are some which are nothing but gauge transforms of $\bar{A}$. These can be fixed by requiring that δA satisfy the gauge fixing condition

$$D_\mu \delta A^\mu = 0 \quad . \tag{2.26}$$

The equations of motion of the fermions imply the following equations on ψ_μ

$$D_\mu \psi_\nu - D_\nu \psi_\mu + \epsilon_{\mu\nu\rho\sigma} D^\rho \psi^\sigma = 0 \quad , \tag{2.27}$$

$$D_\mu \psi^\mu = 0 \quad . \tag{2.28}$$

Thus comparing the two equations given above with equations (2.25) and (2.26) for δA, we find that ψ satisfies the same equations as δA. Thus we can identify

$$\delta A_\mu \leftrightarrow \psi_\mu \quad . \tag{2.29}$$

Let the number of solutions of the equations for ψ_μ be m. Then, a version of the Atiyah-Singer Index theorem can be applied (provided χ and η do not have any zero-modes) and we obtain that

$$m = d(\mathcal{M})$$

Thus, ψ_μ can be identified with the elements of $T^*\mathcal{M}$. The fermion number U is violated by the instanton background by precisely $d(\mathcal{M})$ units. This implies, that only correlation functions which correspond to processes which cause a change of U by precisely $d(\mathcal{M})$ units are non-vanishing. From our identification of ψ_α with $T^*\mathcal{M}$, we can see that the non-vanishing path integrals reduce to top forms on $\mathcal{M}$ This is very much like what we had in the case of the supersymmetric quantum mechanics example we considered in lecture 1.

Case (i): $d(\mathcal{M}) = 0$

When $d(\mathcal{M}) = 0$, the partition function is non-vanishing and gives rise to a topological invariant. The contributions to Z come from isolated instanton solutions. Since there are no zero modes in this problem, there is an exact cancellation between the bosonic and fermionic determinants due to supersymmetry. However, there is an ambiguity with regard to the sign of the answer. The ratio of the determinants is

$$\pm \frac{\prod_i \lambda_i}{\sqrt{|\lambda_i|^2}} = \pm 1 \quad . \tag{2.30}$$

There is no natural way to fix this sign. So one arbitrarily picks a reference instanton and fixes its sign to +1. This uniquely fixes the sign of other instantons, since one can smoothly deform one solution $\bar{A}_1$ to another solution $\bar{A}_2$ by the following combination

$$\bar{A}_t = t\bar{A}_1 + (1-t)\bar{A}_2$$

where $0 \leq t \leq 1$ and require that sign be flipped everytime $\bar{A}_t$ has a zero eigenvalue. Donaldson has shown that there is no obstruction to this procedure. Thus one obtains the partition function to be

$$Z = \sum_i (-)^{n_i} \quad , \tag{2.31}$$

where the sum is over all isolated anti-self dual instanton solutions. This is identical to the result obtained by Donaldson albeit by much simpler arguments.

Case (ii): $d(\mathcal{M}) > 0$

For the case of $d(\mathcal{M}) > 0$, the partition function vanishes and one needs to construct correlation functions (involving Q-closed and gauge invariant observables) which satisfy the ghost number condition to obtain topological invariants. Thus, we need to obtain a gauge invariant observable which is Q-closed. Going back to the commutators of the basic fields in the theory, we see that $[Q, \phi^a] = 0$. However, since ϕ^a is not gauge invariant, the following operators constructed from ϕ^a are gauge invariant as well as Q-closed.

$$\mathcal{O}_{k,0}(x) = \mathrm{Tr}\phi^k \quad , \tag{2.32}$$

where the operator has ghost number $U = 2k$. For $G = SU(n)$, one has $(n-1)$ independent operators. Consider, the action of the derivative on one of these operators

$$\frac{\partial}{\partial x^\mu}\mathcal{O}_{k,0} = k\mathrm{Tr}\phi^{k-1}D_\mu\phi = \{Q, -k\phi^{k-1}\psi_\mu\} \equiv \{Q, \mathcal{O}_{k,1}\} \quad .$$

Thus, the partial derivative of the operator $\mathcal{O}_{k,0}$ is Q-exact and thus decouples from all correlation functions. This is not surprising since correlation functions are independent of the metric and thus independent of the positions of the operators. The above sequence leads to the following set of *descent equations*

$$0 = [Q, \mathcal{O}_{k,0}] \quad , \tag{2.33}$$
$$d\mathcal{O}_{k,0} = \{Q, \mathcal{O}_{k,1}\} \quad , \tag{2.34}$$
$$d\mathcal{O}_{k,1} = [Q, \mathcal{O}_{k,2}] \quad , \tag{2.35}$$
$$d\mathcal{O}_{k,2} = \{Q, \mathcal{O}_{k,3}\} \quad , \tag{2.36}$$
$$d\mathcal{O}_{k,3} = [Q, \mathcal{O}_{k,4}] \quad , \tag{2.37}$$
$$d\mathcal{O}_{k,4} = 0 \quad , \tag{2.38}$$

where we have introduced new objects $\mathcal{O}_{k,r}$ which have ghost number $(2k-r)$ and are r-forms on M. These objects also generate other ways of obtaining topological observables. For example, consider a loop γ in M. Then

$$W_k(\gamma) \equiv \oint_\gamma \mathcal{O}_{k,1} \quad , \tag{2.39}$$

is Q-closed. This follows by using the descent equation to obtain $\{Q, W_k(\gamma)\} = \oint_\gamma d\mathcal{O}_{k,0} = 0$. Similarly, one can consider 2-cycles (surfaces) Σ, 3-cycles on M to obtain observables

$$\begin{aligned} W_k(\Sigma) &= \int_\Sigma \mathcal{O}_{k,2} \quad , \\ W_k(T) &= \int_T \mathcal{O}_{k,3} \quad , \\ \mathcal{L}_k &= \int_M \mathcal{O}_{k,4} \quad . \end{aligned} \tag{2.40}$$

The last of these observables $\mathcal{L}_k$ can be added to the action! The Donaldson invariants for $d(\mathcal{M}) > 0$ are given as follows. Pick homology cycles $\gamma_1, \ldots, \gamma_r$ of dimension $k_1, \ldots, k_r$ respectively. Restricting to the case of $SU(2)$ for simplicity, one obtains the following invariants

$$Z(\gamma_1, \ldots, \gamma_r) = \langle \prod_{i=1} \int_{\gamma_i} W_{k_i} \rangle \quad , \tag{2.41}$$

which is non-vanishing if $\sum_{i=1}^r (4 - k_r) = d(\mathcal{M})$. Thus we have seen how one can construct topological invariants corresponding to the Donaldson invariants. The topological character of the theory is obvious since it satisfied the various conditions we set up at the beginning of the lecture.

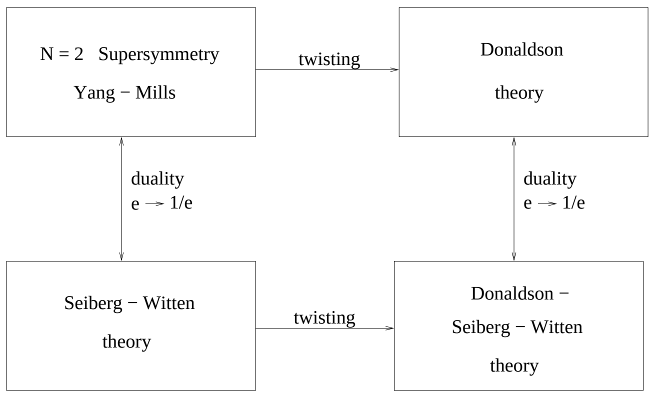
From the mathematical viewpoint, the quantum field theoretic description is not rigourous since path-integrals are not well defined. The understanding of Donaldson's proofs was (and is) formidable even to an accomplished mathematician. It is amusing to see that a physicist can obtain a glimpse of his work in the span of two lectures. However, Donaldson's work preceded the work of Witten and one can claim that QFT has not predicted anything that was not known in this case. However, recent advances due to Seiberg and Witten in the context of N=2 supersymmetric Yang-Mills has provided a dramatic reformulation of Donaldson theory which has provided enormous simplification in a mathematician's understanding of Donaldson's invariants. We shall now briefly describe this progress.

We have seen that the TFT we constructed was such that all correlation functions did not depend on the value of the coupling of constant e. In particular, we chose to take $e \longrightarrow 0$ in order to reduce everything to classical computations. Of course, one could have asked if we could evaluate correlation functions when e was large. This would lead to a strong coupling problem and we would have naively guessed that it would not have been possible to do the computations. Seiberg and Witten have provided a means of doing this calculation. Using a generalisation of electric-magnetic duality[6] for $N = 2$ case, they mapped this strong coupling

[6]Electric-magnetic duality refers to the symmetry of source-free Maxwell's equations under the exchange of the electric and magnetic fields. This simple symmetry can be generalised to include sources. There is strong evidence that theories with $N = 4$ supersymmetry are *self-dual* under the action of a discrete group $SL(2, Z)$ of which electric-magnetic duality forms a Z_2 subgroup.

problem to that of a magnetic monopole interacting with a $U(1)$ gauge field (for the case of $G = SU(2)$. In the weak coupling limit, the appropriate moduli space was the space of anti-self dual instantons. In the strong coupling limit, the moduli space is the space of solutions of a monopole equations i.e., the equations satisfied by a monopole interacting with a $U(1)$ gauge field in a theory with N=2 supersymmetry. The reader is referred to [7, 8] for more details.

The following commutative diagram describes the various relationships.



We hope that these two lectures have given the reader a flavour of what goes into topological field theories. The interested reader can read the review article by Witten[12] for more details as well as references to subsequent papers on the topic.

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