Ricci Flat metrics and the AdS-CFT correspondence

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Plan of talk

- Introduction
- Explicit construction of Ricci-Flat metrics
- The Leigh-Strassler deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory
  - A conjecture on the spectrum of chiral primaries
  - Searching for the gravity dual to the CFT

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- The Leigh-Strassler deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory
  - A conjecture on the spectrum of chiral primaries
  - Searching for the gravity dual to the CFT

Credits: Aswin K. Balasubramanian, Pramod Dominic, Chethan N. Gowdigere, Hans Jockers, K. Madhu.

String Theory @ IITM: SG and Prasanta K. Tripathy

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Introduction

The AdS-CFT correspondence

- In its original form, it relates type IIB strings propagating in a ten-dimensional spacetime $AdS_5 \times S^5$ to a four-dimensional conformal field theory (CFT): $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. [Maldacena 1997]
- It is a strong-weak duality – strong coupling in string theory gets mapped to weakly coupled CFT and vice versa.
- It makes it hard to verify but if true provides a nice way to carry out computations at strong coupling.

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- It makes it hard to verify but if true provides a nice way to carry out computations at strong coupling.
- The duality that relates the Ising model at high-temperature to the Ising model a low-temperature is a similar example – it predicts a phase transition though one cannot obtain the critical temperature.

What is Ricci-Flatness?

- A metric on a manifold assigns a length to any curve connecting a pair of points. Two nearby points with separation dx^i ($i = 1, 2, \ldots, d$) are assigned a distance ds defined by

$$ds^2 = g_{ij}(x)\ dx^i dx^j \ ,$$

  We usually refer to g_{ij} as the metric.

- Is it possible to find coordinates such that $g_{ij} = \delta_{ij}$ everywhere? In general, the answer is no.

- The best one can do in the neighbourhood of a point is a Taylor series of the form

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}\ x^k x^l + \mathcal{O}(x^3)\ .$$

  The obstruction is called the Riemann curvature tensor.

What is Ricci-Flatness?

- The Ricci tensor is a second-rank symmetric tensor obtained from the Riemann curvature tensor by contracting a pair of indices $R_{ij} \equiv g^{kl} R_{ikjl}$.
- Einstein's equations in general relativity is written in terms of this tensor ($R \equiv g^{ij} R_{ij}$)

$$R_{ij} - \frac{1}{2} R\, g_{ij} = 8\pi G_N\, T_{ij}\ .$$

- A manifold is called Ricci-Flat(RF) if $R_{ij} = 0$ at all points.
- In 3 dimensions, the vanishing of the Ricci tensor implies the vanishing of the Riemann curvature tensor. This is not true in 4 and higher dimensions.

Appearance of RF manifolds: String Theory

- Superstring theory requires spacetime to be ten-dimensional. However, spacetime as we perceive it at low energies is four-dimensional.
- A simple and effective way to get around this is to assume that six of the dimensions are compact and small enough to be invisible at low energies.
- String compactification assumes that spacetime is assumed to be of the form $\mathbb{R}^{1,3} \times M^6$, where M is a compact six- dimensional manifold.
- Consistency of string propagation (conformal invariance) requires M to be Ricci-Flat to leading order.

AdS-CFT and Ricci-Flat manifolds

- Non-compact Ricci-Flat manifolds make an appearance in the context of the AdS-CFT correspondence (more generally, the gravity-gauge correspondence).
- This correspondence more generally relates type IIB string theory on a spacetime $AdS_5 \times X^5$ to a four-dimensional superconformal field theory (CFT).
- Let M^6 be a non-compact six-dimensional manifold obtained as a cone over X^5 i.e., consider a six-dimensional metric obtained from the five-dimensional metric on X (which we write as ds^2_X):

$$ds^2_M = dr^2 + r^2\, ds^2_X \quad , \; r \in [0, \infty) \; .$$

  When $X = S^5$, $M = \mathbb{R}^6 = \mathbb{C}^3$.
- Consistency of string propagation on $AdS_5 \times X^5$ translates into the condition that M be Ricci-Flat. .

Ricci-Flow

- Consider the dynamical system, called the Ricci-Flow (due to Richard Hamilton)

$$\frac{dg_{ij}}{dt} = -R_{ij} \ ,$$

  where g_{ij} are the components of the metric.

- Ricci-Flat metrics appear as the fixed-points of the dynamical system This is an area being actively pursued in mathematics.
- The proof of the Poincaré conjecture by Perelman makes use of this dynamical system.

Explicit Ricci-Flat metrics six-dimensional manifolds

with Aswin Balasubramanian and Chethan Gowdigere

Symplectic potentials and resolved Ricci-Flat ACG metrics.

Aswin K. Balasubramanian, SG, Chethan N. Gowdigere.

Classical and Quantum Gravity 24 (2007) 6393-6415 [arXiv:0707.4306] [hep-th]

Kähler manifolds

- Kähler manifolds admit symplectic and complex structures that are compatible. For our purposes, it suffices to know that the metric is determined completely in terms of derivatives of a single function – schematically one has $g_{ij} \sim \partial_i \partial_j G(x)$.
- The condition of obtaining the RF metric reduces to solving a non-linear partial differential equation for the function. This is typically a hard problem.
- In the context of non-linear DE's such as the KdV equation, solutions have been found when there is an integrable structure.
- Is there such a structure underlying finding RF Kähler manifolds?

Hamiltonian two-forms

- Apostolov, Calderbank and Gauduchon (ACG) observed that if a Kähler manifold admits a Hamiltonian two-form, then there exists a special set of coordinates.
- In these coordinates, the metric takes a special form where the single function of several variables, $G(x)$ gets replaced by several functions of one variable.
- This reduces the problem to solving several ODE's. In fact, Ricci Flatness is easier to impose.
- The neat result is that ACG provide a classification of metrics that admit a Hamiltonian two-form. In six-dimensions, such metrics are parametrised by an additional label, $\ell = 1, 2, 3$. Thus, there are three families of such metrics.

The $\ell = 3$ ACG metric

The special coordinates are (ξ, η, χ, t_1, t_2, t_3). The metric in these coordinates is (with $\Delta = (\xi - \eta)\,(\eta - \chi)\,(\chi - \xi)$)

$$
\begin{aligned}
ds^2 = &- \Delta \left[\frac{d\xi^2}{(\eta - \chi) f(\xi)} + \frac{d\eta^2}{(\chi - \xi) g(\eta)} + \frac{d\chi^2}{(\xi - \eta) h(\chi)} \right] \\
&- \frac{1}{\Delta} \Big[(\eta - \chi)\, f(\xi)\, (dt_1 + (\eta + \chi) dt_2 + \eta\, \chi\, dt_3)^2 \\
&\quad + (\chi - \xi)\, g(\eta)\, (dt_1 + (\chi + \xi) dt_2 + \chi\, \xi\, dt_3)^2 \\
&\quad + (\xi - \eta)\, h(\chi)\, (dt_1 + (\xi + \eta) dt_2 + \xi\, \eta\, dt_3)^2 \Big] \, .
\end{aligned}
$$

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& - \frac{1}{\Delta}\Big[(\eta-\chi)\,f(\xi)\,(dt_1 + (\eta+\chi)dt_2 + \eta\,\chi\,dt_3)^2 \\
& \quad + (\chi-\xi)\,g(\eta)\,(dt_1 + (\chi+\xi)dt_2 + \chi\,\xi\,dt_3)^2 \\
& \quad + (\xi-\eta)\,h(\chi)\,(dt_1 + (\xi+\eta)dt_2 + \xi\,\eta\,dt_3)^2\Big]\,.
\end{aligned}
$$

The scalar curvature is given by

$$
R = -\frac{f''(\xi)}{(\xi-\eta)(\xi-\chi)} - \frac{g''(\eta)}{(\eta-\xi)(\eta-\chi)} - \frac{h''(\chi)}{(\chi-\eta)(\chi-\xi)}
$$

Our analysis

- We started with the three families of ACG metrics and carried out a global analysis by identifying the associated polytope.
- This global analysis enabled us to identify metrics for specific choices of the functions – it turns out that we always need $f/g/h$ to be cubic polynomials.
- All known examples of RF metrics appear in this class!
- In the $\ell = 1$ class, we obtained a new set of metrics. These correspond to a partial resolution of some well-known singular spaces, these are cones over five-dimensional spaces called $Y^{p,q}$ that appeared in the context of the AdS-CFT correspondence.

  [Gauntlett-Martelli-Sparks-Waldram]
- Our analysis is not exhaustive and there might be more!

The Leigh-Strassler deformation of $\mathcal{N} = 4$ SYM

(i) The spectrum of chiral primaries

with K. Madhu and Pramod Dominic

Chiral primaries in the Leigh-Strassler deformed N=4 SYM - a perturbative study.
Kallingalthodi Madhu & SG

JHEP 05 (2007) 038. [hep-th/0703020]

The LS deformation of $\mathcal{N}=4$ SYM

- The original AdS-CFT correspondence involves a CFT with a high degree of supersymmetry.
- Leigh and Strassler(LS) argued that this CFT admitted a two-parameter set of deformations that reduced supersymmetry to the minimal $\mathcal{N}=1$.
- The LS theory has the same fields as in $\mathcal{N}=4$ SYM theory. It contains one $\mathcal{N}=1$ vector multiplet and three chiral multiplets that we will denote by Φ_1, Φ_2, Φ_3, each of which transform in the adjoint of $SU(N)$ (not $U(N)$).
- The superpotential for $\mathcal{N}=4$ SYM theory is

$$W_0 = h\ \mathrm{Tr}\left(\Phi_1\left[\Phi_2, \Phi_3\right]\right) = h\ \mathrm{Tr}\Big(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2\Big)$$

The LS superpotential

- The superpotential for the LS theory is of the form

$$W = W_0 + \frac{1}{3!} c^{ijk} \, \mathrm{Tr} \left(\Phi_i \Phi_j \Phi_k \right) \ ,$$

  where c^{ijk} is totally-symmetric in its indices.
- It is useful to think of the three chiral fields as complex coordinates on $\mathbb{C}^3$.
- c_{ijk} has 10 independent components, using simple linear redefinitions acting on the fields ($SL(3,\mathbb{C})$ acting on the three fields), we find only two non-trivial deformations. These are the two marginal deformations of Leigh and Strassler.

Chiral Primaries in CFT

- There exist a special class of operators in the CFT whose scaling dimension remains protected from quantum corrections (called ‘anomalous dimensions’).
- There exist a special class of operators with vanishing anomalous dimension that are called chiral primaries.
- Due to supersymmetry, several aspects of these operators at strong coupling can be studied as well. So it important to understand all such operators.
- We studied the spectrum of the (single-trace) chiral primaries by perturbatively computing the anomalous dimension of single-trace operators upto and including dimension six and looking for operators for which the anomalous dimension vanished.

The conjecture

- Using a discrete non-abelian symmetry in theory given by the trihedral group $\Delta(27)$, we were able to classify the protected operators which lead to a sharp conjecture for arbitrary dimensions.
- When the dimension $\Delta_0 > 2$, we conjectured [SG, Madhu]

Scaling dim.	$\Delta_0 = 3r$	$\Delta_0 = a$ mod 3
$\mathcal{N} = 4$ theory	$\mathcal{L}_{0,0} \oplus \frac{r(r+1)}{2} \left[\oplus_{i,j} \mathcal{L}_{i,j} \right]$	$\frac{(\Delta_0+1)(\Delta_0+2)}{6} \mathcal{V}_a$
β-def. theory	$\mathcal{L}_{0,0} \oplus_j \mathcal{L}_{0,j}$	$\mathcal{V}_a$
LS theory	$2\ \mathcal{L}_{0,0}$	$\mathcal{V}_a$

$\mathcal{L}_{i,j}$ $(i, j = 0, 1, 2)$, $\mathcal{V}_1$ and $\mathcal{V}_2$ are irreps of $\Delta(27)$.

- The conjecture was based on explicit computations up to and including $\Delta_0 = 6$.

Towards proving the conjecture

- The conjecture has not been verified when $\Delta_0 > 6$ – direct calculations become very hard.
- At $\Delta = 6$, the computation involves 26 operators that mix quantum mechanically and the anomalous dimensions are given by the eigenvalues of a 26×26 matrix – this after using all symmetries else it would have been 58 dimensional.
- So we are pursuing a different approach. It has been shown that the one-loop anomalous dimensions of all chiral operators can be obtained as the spectrum of a one-dimensional supersymmetric spin-chain.
- The length of the spin-chain is mapped to Δ_0.

The spin-chain

- Due to supersymmetry, the energy eigenvalues of the Hamiltonian are bounded from below by zero.
- The vanishing of the anomalous dimensions gets mapped to a statement of number of eigenvectors that have eigenvalue zero.
- We are currently using the trihedral symmetry to organise the computation and we hope to prove the conjecture, at the very least, for spin-chains of length $3L$. [SG. Pramod Dominic]
- A much harder problem is to obtain the full spectrum of the spin-chain. Note that the spin chain has 3^N states where N is the length of the chain.
- Is there a Bethe ansatz for this spin-chain? Yes, for the β-deformation.

Searching for the gravity dual to the LS theory

with Chethan Gowdigere

Effective superpotentials for B-branes in Landau-Ginzburg models.
SG and Hans Jockers.

JHEP 10 (2006) 060. [hep-th/0608027]

The gravity dual for LS theory

- While it is anticipated that the Leigh-Strassler theory will be dual to strings moving on a spacetime that is $AdS_5 \times X^5$ for some X, the precise space X has not been determined!
- Recall that the LS theory was obtained as a deformation of $\mathcal{N} = 4$ SYM theory for which $X = S^5$.
- So the naive expectation is that S^5 should admit two deformations that are compatible with conformal invariance of string theory – i.e., a cone over S^5 which is nothing but $\mathbb{R}^6$ should admit deformations. There are none!
- Is there a problem? However, the answer is known for a one-parameter deformation, the β-deformation.

[Lunin-Maldacena]

Generalising Ricci-Flatness

- In the context of string theory, the metric is one of several fields in theory.
- For instance, there is a second-rank antisymmetric tensor, called the B-field, $B = \frac{1}{2} B_{ij} dx^i \wedge dx^j$, a scalar called the dilaton, Φ, that is common to all string theories.
- There are also other p-form gauge fields, $C^{(p)} = \frac{1}{p!} C_{i_1 \ldots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, that appear.
- So the condition for conformal invariance gives rise to a much more complicated system of coupled partial differential equations involving all these fields.

Generalising Ricci-Flatness

- In the limit that these fields vanish or take constant values, on recovers the condition that the metric must be Ricci-Flat.
- The term manifolds is typically used for spaces that are Riemannian and thus have a metric.
- A generalised manifold can be defined to be a space with the various massless fields of string theory.
- A generalised geometry on them is given by imposing the conditions for conformal invariance of string theory.

- The LS superpotential appeared in a computation of mine in a diffferent context – this might give us a clue to finding the precise background. [SG, Jockers]

Conclusion

I hope I have given you a flavour on some of the problems I am working on.

THANK YOU