Derived Categories - I

The McKay correspondence

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- w/ Bobby Ezhuthachan & T. Jayaraman
(to appear)

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What is a D-brane?
(Dirichlet)

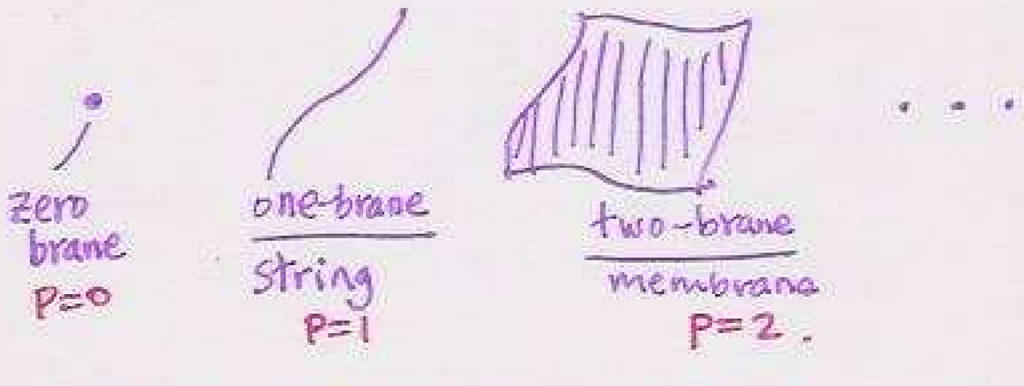
Def 1 Supersymmetric solitons in type II/I string theory carrying RR charges.

Def 2 Extended objects/defects (P-brane
on which open strings can end
"boundary cond" "

in 10 dim

Possible p-values

IIA	0	2	4	6	8	
IIB	-1	1	3	5	7	9



Both defs. have distinct regimes of validity

Matching them leads to
open-closed duality
e.g. AdS-CFT correspondenc

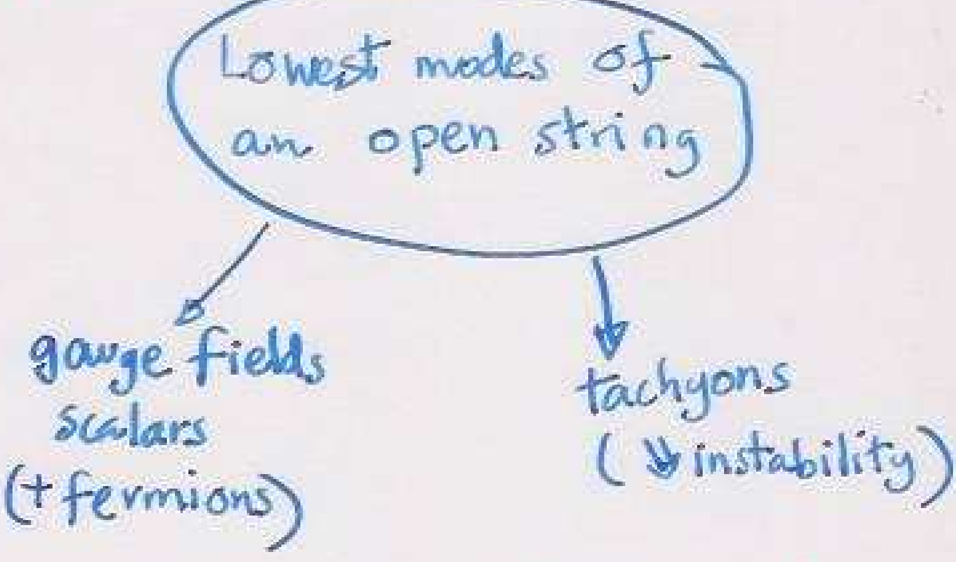
Excitations of D-branes

Polchinski The collective modes of a "single" D-brane

$\parallel$

modes of an open-string starting and ending on the D-brane.

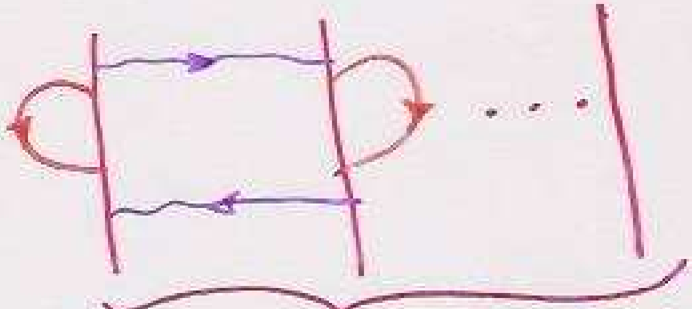
- For "multiple" D-branes, additional mode arise from open-strings connecting distinct D-branes.



Lowest modes of an open string

gauge fields
scalars
(+ fermions)

tachyons
(⇓ instability)



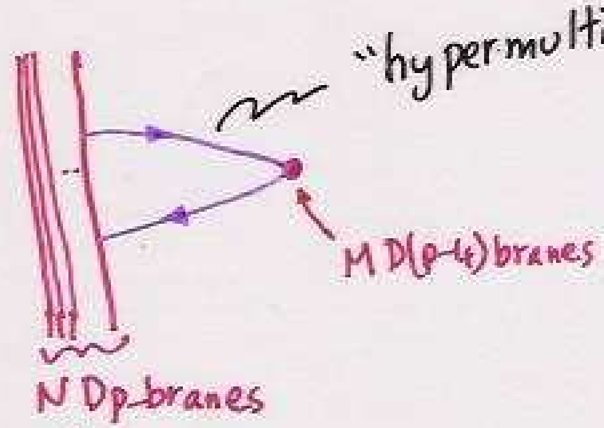
N-parallel Dp-branes

"Bosonic spectrum"

- U(N) gauge field
- (9-P) adjoint scalars

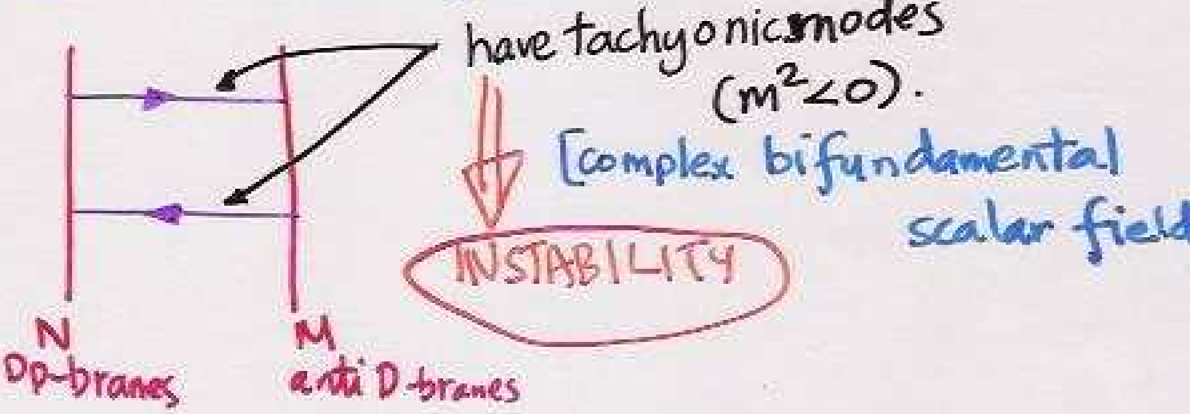
(+ fermions)

dim. redn of $\mathcal{N}=1$ SY
from 10 dim to
(P+1) dim.

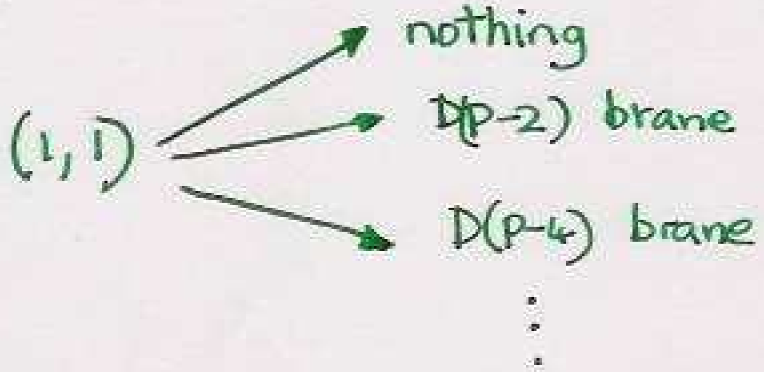
- 

. The (Dp-4) brane can "dissolve" into the Dp-branes — appears as an INSTANTON!
(ADHM construction!)

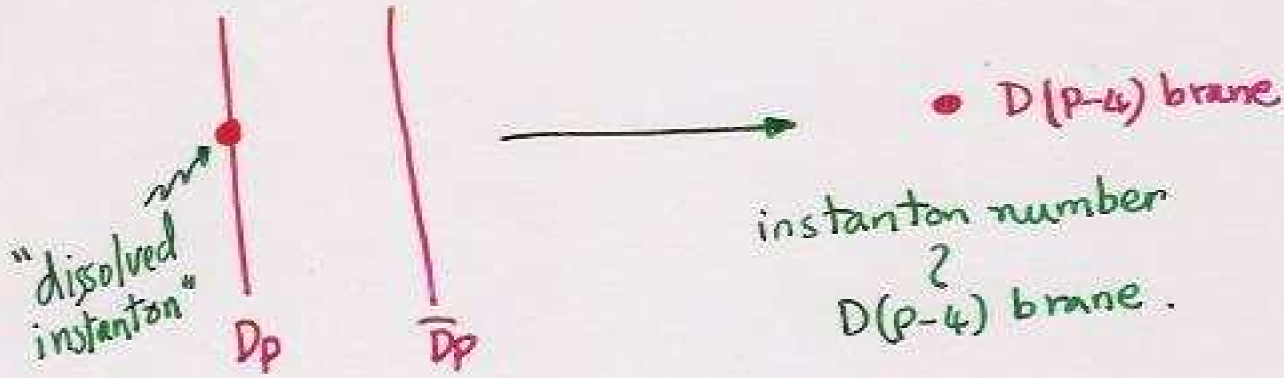
- Brane - Antibrane system.



<u>Ashoke Sen:</u> The tachyonic instability ⇒ "tachyon condensation" with several possibilities



Why are there so many possible end points?



Sen: (N, M) D9-$\overline{\text{D9}}$-brane systems (resp. D8-$\overline{\text{D8}}$) in type IIA (resp. type IIB) generate all lower-dimensional branes via tachyon condensation.

So far, all examples have been in flat 10-dimensional space.

Other compactifications?

Working Example

X = six-dimensional Ricci-flat Kähler manifold. (CY3)

type II string compactified on X

?

[effective] 4-dimensional theory.

Ideas from tachyon condensation in flat space led to several conclusion in this case!

D-brane charges⊛ are classified by K-theo
II A (resp II B) ⟶ $K^0(X)$ [resp. $K^1(X)$

Simpler statement when X has no torsion.

$$K^0(X) \sim \sum_p H^{2p}(X)$$

$$K^1(X) \sim \sum_p H^{2p+1}(X)$$

D2p branes wrapping 2p-dim. cycles of X

D(2p+1) branes wrapping (2p+1)-dim cycles in X.

⊛ Restrict to particles in $R^{3,1}$

Classical Geometry of X

X — 3 cycles $(2h_{2,1}+2)$ such cycles.

X — 0, 2, 4, 6 cycles

$h_{1,1}$ such cycles

Particles in (3+1) dim spacetime can arise in several ways.

e.g. D3-branes wrapping 3 cycles of X.

$$\vec{Q} \equiv (\underbrace{Q^e_0, Q^e_1, \ldots, Q^e_{h_{2,1}}}_{\text{electric}} ; \underbrace{Q^m_0, Q^m_1, \ldots, Q^m_{h_{2,1}}}_{\text{magnetic}})$$

CHARGE VECTOR

possible charges of such particles

$$(A_0, A_1, \ldots, A_{h_{2,1}} ; B_0, B_1, \ldots, B_{h_{2,1}})$$

integral basis for 3cycles on X.

two such particles

$$\boxed{\mathcal{I}(E_1, E_2) = \vec{Q}_1 \cdot \Omega \cdot \vec{Q}_2 \in \mathbb{Z}}$$

Dirac quantisatio

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_{\overline{h_{2,1}+1} \times \overline{h_{2,1}+1}} \\ \mathbb{1} & 0 \end{pmatrix}$$

$h_{2,1}+1$

this is also the geometric intersection of the cycles!

Masses of these particles

"BPS formula"

$$\underset{\substack{\nearrow \\ \text{central charge}}}{Z}(\vec{Q}) \equiv \vec{Q}\cdot \underset{\substack{\nwarrow \\ \text{periods of } X\text{.}}}{\vec{\Pi}}\left(\substack{\text{some}\\ \text{moduli}}\right)$$

$$\boxed{M \sim |Z(\vec{Q})|}$$

- Π^s are solutions of differential equations (of the GKZ type).

  $\Rightarrow$ can analytically continue to all values of moduli [Candelas et al.]

- Monodromy about reg. sing. points

<u>eg.</u>

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}\right)f=0 \qquad \begin{array}{l} z=0 \\ \text{reg. sing.} \\ \text{point.}\end{array}$$

has two solutions $f_0=1$; $f_1=\ln z$.

$$z \to z e^{2\pi i}$$

$$\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix}}_{\substack{\text{Monodromy}\\ \text{matrix}}}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

Derived Categories – a motivation.

$\vec{Q}$ – possible D-brane charge

K-theory classes distingoishes objects with distinct charge vectors.

Are there more structures ?

central cha

$$\text{grade of a brane} \equiv \underset{\text{fn of moduli}}{\varphi}(\vec{Q}) = \frac{1}{\pi}\mathrm{Im}(\log Z) \quad \text{mod } 2$$

Doug

(Recall, multivaluedness of periods)

Extend to all values of moduli via analytic continuation after fixing φ for all branes at a particular value of moduli.

More or less forces one to let the grade take values in $\mathbb{R}$. [since periods and hence phase of Z will also be multivalued].

Remark: $-\vec{Q}$ = charge of antibrane

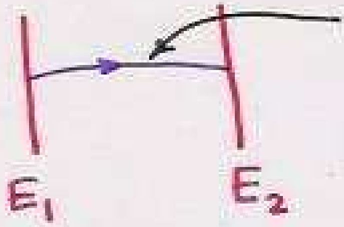
$$\Rightarrow Z(-\vec{Q}) = -Z(Q)$$

$$\Rightarrow \boxed{\varphi(-\vec{Q}) = \varphi(\vec{Q}) + 1}$$

Shift of grade ! takes brane $\Leftrightarrow$ antibrane.

D-Brane decay

Unlike flat space D-branes, there is the possibility of decay of D-branes for CY3 compactifications (like $\mathcal{N}=2$ SYM in 3+1)



mass of the lowest mode varies with moduli.

At some points, m^2 switches sign from tachyonic to massive.

bound state $[E_1 \to E_2] \xrightarrow[\text{decay}]{\text{can}} E_1 + E_2$.

(across lines of magginal stability).

$m^2 < 0$ | $m^2 > 0$

A related mathematical construct

$\mathbb{C}P^1 \sim S^2$: • $|\phi_1|^2 + |\phi_2|^2 = r$

• gauge equivalence $\phi_i \sim e^{i\theta} \phi_i$

$m^2=-$

$r \gg 0$

Procedure gives $\mathbb{C}P^1$

$r \ll 0$

has no solutions.

The derived category is a mathematical construct that

(i) is a refinement over K-theory
distinguishes zerobranes at diff. loc

(ii) captures the analog of the equivalence

$$(N+m, M+m) \sim (N, M)$$

<u>if</u> $(m, m) \rightarrow$ nothing.

i.e., $(m,m) \rightarrow$ nothing is like the identity operator. <u>e.g.</u> $\mathbb{Z}_+$ semi-group under addition

(iii) incorporates the gradings of the D-branes.

grade of the brane = position in a complex

Plan of the talk

1. An extended introduction (just finished)
2. (a) D-branes on $\mathbb{C}^3/\mathbb{Z}_3$ and its resolution.
   (b) The McKay Correspondence.
3. D-branes on the quintic
4. The story of the missing D0-brane.
5. The resolution (by 3 investigators)
6. Conclusion + Outlook.

Quantum
McKay
correspondence

— α —

Basic Examples of CY3

① $X_3 =$ resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$.

(non-compact)

$$h^c_{1,1}(X) = 1$$

② $Q =$ quintic hypersurface in $\mathbb{C}P^4$.

(compact)

$$h_{1,1}(Q) = 1 \quad ; \; h_{2,1}(Q) = 101 .$$

$$\boxed{G(\phi_1, \phi_2, \cdots, \phi_5) = C_{i_1 i_2 i_3 i_4 i_5} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_5} = 0}$$

ϕ_i = homogeneous coordinates ~~is~~ for $\mathbb{C}P^4$.

$$\phi_i \sim \lambda \, \phi_i \qquad \lambda \in \mathbb{C} - \{0\} \equiv \mathbb{C}^*$$

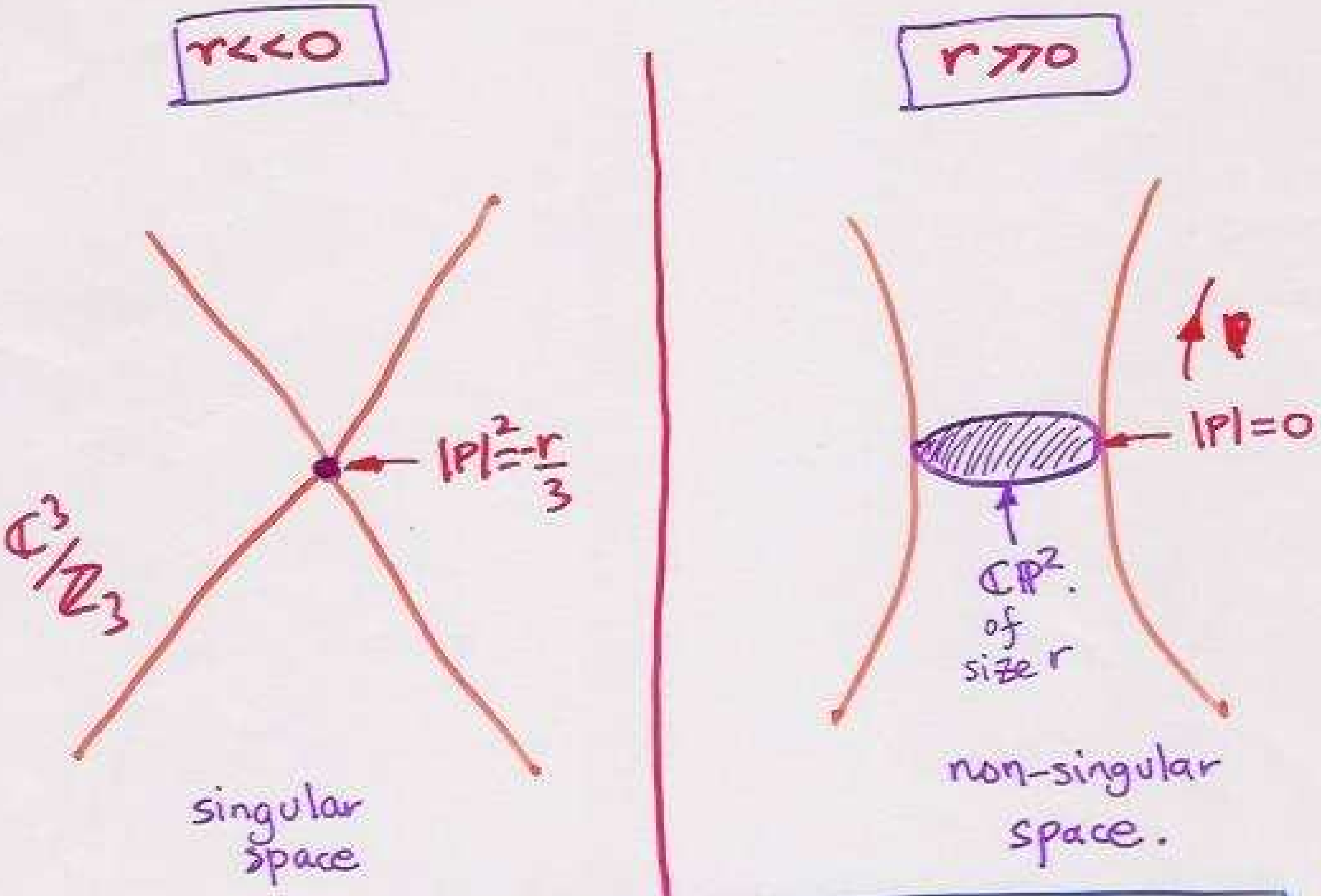
We shall focus on D-branes wrapping even-dim (holomorphic) cycles of X_3 or Q. for the rest of the talk. "B-branes"

More structure of X_3.

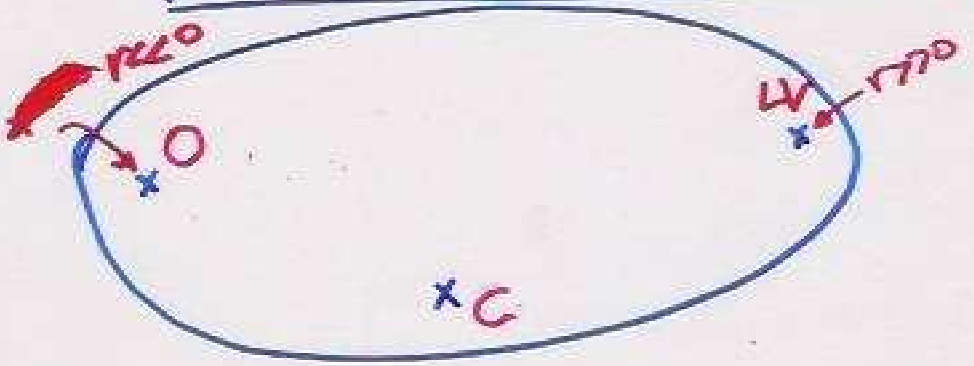
$$|\phi_1|^2+|\phi_2|^2+|\phi_3|^2-3|P|^2=r$$

gauge equivalence: $\phi_i \sim e^{i\theta}\phi_i$

$P \sim e^{-3i\theta}P$



KAHLER MODULI SPACE OF X



O; C; V: Regular sing. points of the ODE satisfied by PERIODS.

B-branes on X_3

- At O [the orbifold point].
  - Zero branes localised at the singularity.
  - There are three kinds of such branes

$$\boxed{S_1, S_2, S_3}$$

"fractional zero-branes"

basis branes.

  - carry $\frac{1}{3}$ units of zero-brane charge (in terms of flat space)
    ⇒ cannot be moved away from sing.

- At LV [large volume].
  - Focus on branes on $\mathbb{CP}^2$ (that the singularity has become).
  - B-branes ↔ vector bundles/coherent sheaves on $\mathbb{P}^2$.

grades

* D4 brane wrapping $\mathbb{P}^2$ $\qquad \mathcal{O}_{\mathbb{P}^2}$

* D2 brane wrapping $\mathbb{P}^1 \subset \mathbb{P}^2$ $\qquad [\overset{0}{\mathcal{O}_{\mathbb{P}^2}} \to \overset{1}{\mathcal{O}_{\mathbb{P}^2}(1)}]$

* D0 brane on a point in $\mathbb{P}^2$. $\qquad [\overset{-1}{\mathcal{O}(-1)} \to \overset{0}{\mathcal{O}} \to \overset{1}{\mathcal{O}(1)}]$

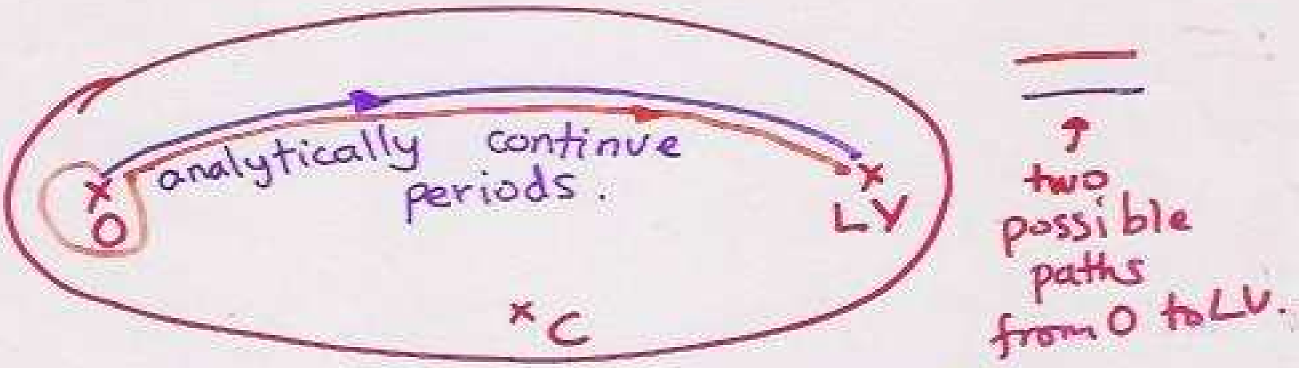
Relating B-branes at O and LV

Basis branes at O: S_1, S_2, S_3.

Basis branes at LV:
$$\begin{array}{ccc}\mathcal{O}_{\mathbb{P}^2}(-1), & \mathcal{O}_{\mathbb{P}^2}, & \mathcal{O}_{\mathbb{P}^2}(1)\\ \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$}\\ R^1 & R^2 & R^3\end{array}$$

Strategy • Identify periods for S_1, S_2, S_3 at O.

Diaconescu - Gomis
Douglas-Fiol-Römel..



• Identify periods for R^1, R^2, R^3 and compare with the analytically continued periods for S_1, S_2, S_3.

Subtlety: Multivaluedness ⇒ result is path-dependent.

e.g. monodromy at O has the following action

$$\begin{tikzcd} M_0 : S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[ll, bend left] \end{tikzcd}$$

Completing the Matching

"The McKay Correspondence"

The intersection matrix

$$\mathcal{I}(S_i, R^j) = \delta_i^{\ j}$$

Rewrites in Russian notation (!)

$$\langle S_i \,|\, R^j \rangle = \delta_i^{\ j}$$

to emphasise that the R and S bases are **dual** to each other.

In mathematical terms; the R^s are a basis for $K(X)$ and the S^j are a basis for $K^c(X)$.

Ito
Nakajima

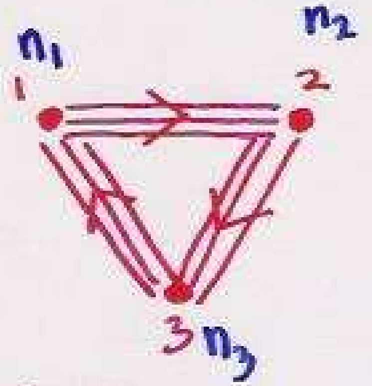
Recall, S_i^s were obtained at the orbifold point. It has information about the resolved space.

Aside: Classical McKay Correspondence

$\mathbb{C}^2/\Gamma_{ADE}$ ⊸ resolution. has $\mathbb{P}^1$'s whose intersection matrix ⟺ Cartan matrix for ADE.

Quivers from basis branes

$$I(S_i,S_j)=\begin{pmatrix}0 & -3 & 3\\ & 0 & -3\\ & & 0\end{pmatrix}$$



$\underset{i}{\bullet}$ vertex ~ S_i

$i \to j$ - number of arrows ~ $|I(S_i,S$

sign I = directi

(McKay) Quiver

captures the data in the intersection matrix

analog of the Dynkin diagrams

encodes the worldvolume spectrum of the D-brane (n_1, n_2, n_3)

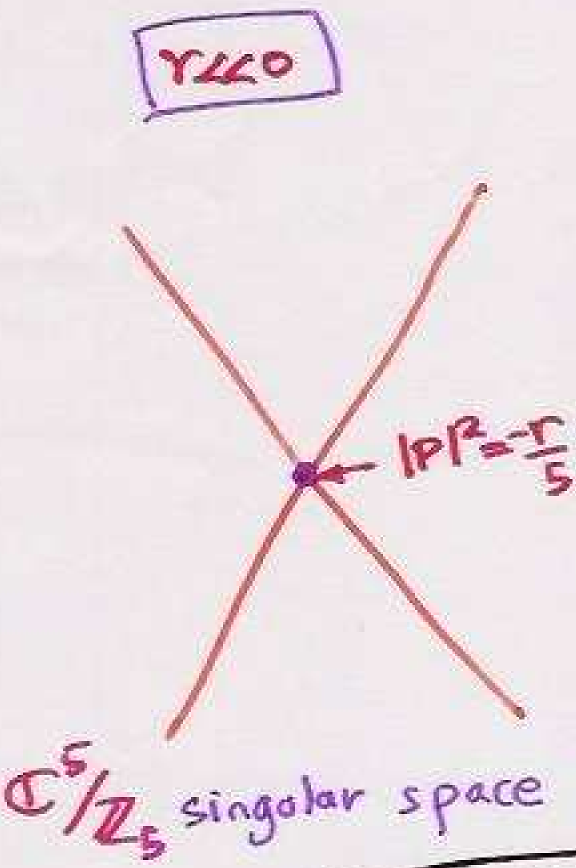
- $U(n_1)\times U(n_2)\times U(n_2)$ vector mult.
- $3(n_1,\bar{n}_2)$; $3(n_2,\bar{n}_3)$, $3(n_3,\bar{n}_1)$ chiral multiplets.

Onto our second example: Q.

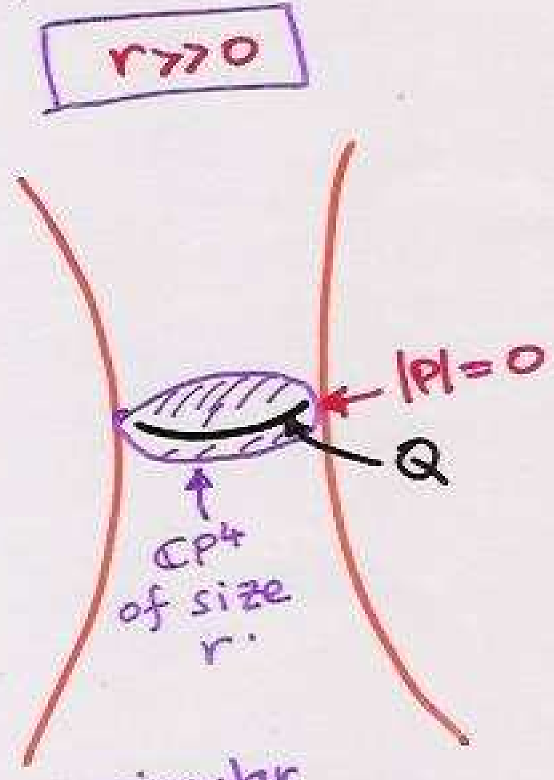
X_5: $|\phi_1|^2+|\phi_2|^2+|\phi_3|^2+|\phi_4|^2+|\phi_5|^2-5|p|^2=$

gauge equivalence: $\phi_i \sim e^{i\theta}\phi_i$

$p \sim e^{-5i\theta} p$.

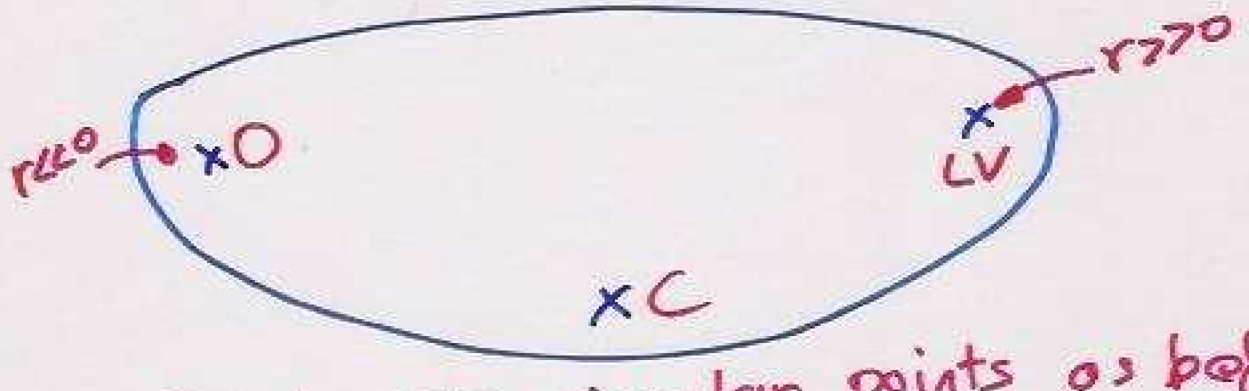


$\mathbb{C}^5/\mathbb{Z}_5$ singular space



non-singular space.

KAHLER MODULI SPACE OF X



O, C, V = reg. singular points as before

B-branes on X_5

- At O [the orbifold point].
  - zero-branes localised at the sing.
  - There are five fractional zero-branes

$$\boxed{S_1, S_2, S_3, S_4, S_5}$$

basis branes for $\mathbb{P}^4$.

  - carry $\frac{1}{5}$ units of D0-brane charge.

- At LV.
  - Focus on the blowup of the singularity i.e., $\mathbb{CP}^4$.

* D8-brane wrapping $\mathbb{P}^4$ $\quad \mathcal{O}$

* D6-brane wrapping $\mathbb{P}^3 \subset \mathbb{P}^4$ $\quad [\overset{0}{\mathcal{O}} \to \overset{1}{\mathcal{O}(1)}]$

* D4-brane wrapping $\mathbb{P}^2 \subset \mathbb{P}^4$ $\quad [\overset{-1}{\mathcal{O}(-1)} \to \overset{0}{\mathcal{O}(} \to \overset{1}{\mathcal{O}(1)}]$

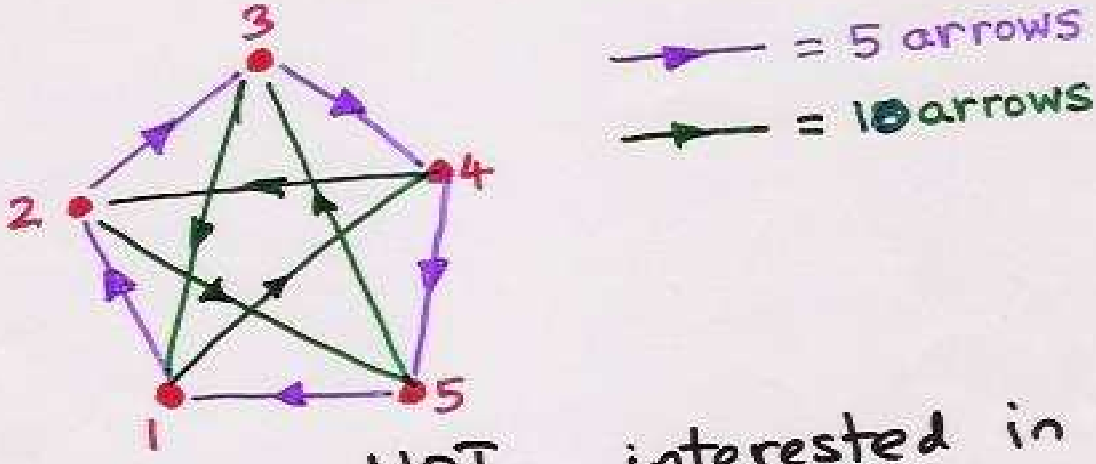
* D2-brane wrapping $\mathbb{P}^1 \subset \mathbb{P}^4$ $\quad [\overset{-1}{\mathcal{O}(-1)} \to \overset{0}{\mathcal{O}} \to \overset{1}{\mathcal{O}(1)} \to \overset{2}{\mathcal{O}(2}$

* D0-brane in a pt. $\subset \mathbb{P}^4$. $\quad [\overset{-2}{\mathcal{O}(-2)} \to \cdots\cdots \to \overset{2}{\mathcal{O}(2)}$

Again, we have the generalised McKay corresp. for $\mathbb{P}^4$

$$\langle S_i | R^j \rangle = \delta_i^j$$

with the quiver



But, we are NOT interested in B-branes on $\mathbb{P}^4$, rather we want B-branes on Q, i.e., a hypersurface in $\mathbb{P}^4$.

A first guess:

Restrict S_i to Q. — $S_i|_Q$.

$$\boxed{V_i \equiv S_i|_Q}$$

Does this work?

Almost!

Can show

$$\boxed{V_1 + V_2 + V_3 + V_4 + V_5 = \text{nothing}}$$

- Choose V_1, V_2, V_3, V_4 as basis branes (since $\bar{V}_5$ can be obtained as $V_1 + V_2 + V_3 + V_4$).
- Leads to interesting predictions for ground states of various combinations of V_i. (some not all verified)

  Douglas, SG,
  Jayaraman,
  Tomasiello
- Appearance of nontrivial mathematical structures (A_∞-algebras,
- Do the basis branes V_1, V_2, V_3, V_4 suffice? Do they generate all charges appearing in $K_\circ^\circ(Q)$?

A problem: Where is the D0 brane?

Analysis of the D-brane charges show that the best one can do is get an object with charge of 5 D0-branes.

The same happens for D2-branes — again can get one with the charge of 5 D2-branes.

The more precise statement is that one obtains an order 25 sub-lattice in the full lattice of D-brane charges.

To reiterate,

Where is the zero brane?

Finding the zero-brane

"Quantum McKay Correspondence"

A digression

The point 0 in the quintic moduli space has two descriptions.

- Gepner model ← Recknagel-Scho
- Landau-Ginzburg Orbifold.

SG, Jayaraman, Sarkar 99

D-brane in these models were studied independently.

Possible b.c's in the LG orbifold

- $\phi_1 = \phi_2 = \cdots = \phi_5 = 0$ ← same as fractional zerobranes for $\tilde{\mathbb{C}}^5/\mathbb{Z}$
- $\boxed{(\phi_1 + \phi_2) = 0 \, ; \, \phi_3 = \phi_4 = \phi_5 = 0 \quad \text{when } G = \phi_1^5 + \phi_2^5 + \cdots + \phi_5^5 .}$

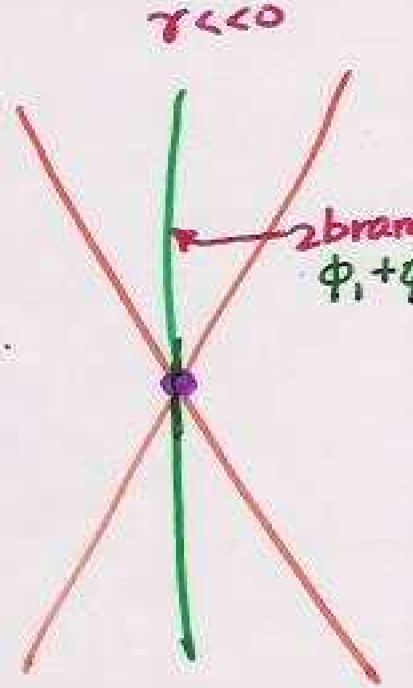
⋮

Alternate/equivalent(?) sol^n – Ashok, Diaconescu et al.

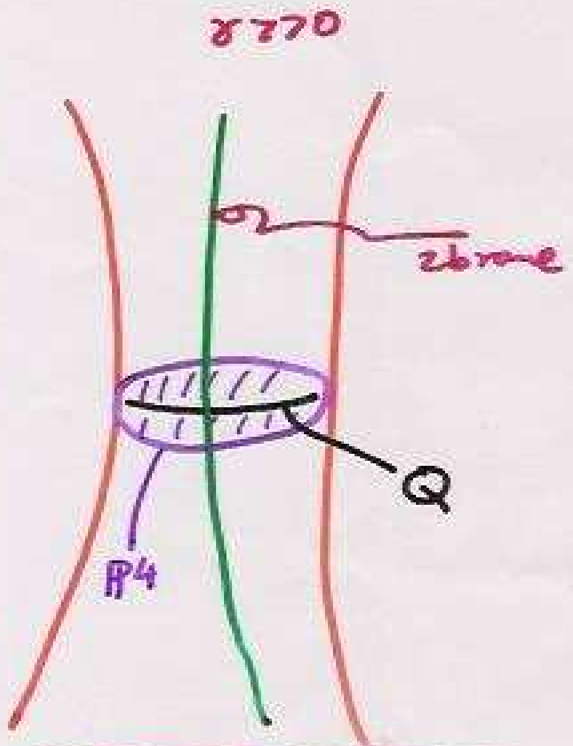
A proposal: These fractional two-branes give rise to the D0-brane!

SG
Jayaraman
&
Bobby
Ezhuthachan

Fractional 2-branes



Picture at O



Picture at LV

- We Find five fractional two branes (on $\mathbb{C}^5/\mathbb{Z}_5$) $S_1^{(2)}, S_2^{(2)}, \dots, S_5^{(2)}$

- $\mathcal{I}(S_i^N, S_j^N)\big|_Q$

$$= \begin{pmatrix} 0 & 1 & -3 & 3 & -1 \\ -1 & 0 & 1 & -3 & 3 \\ 3 & -1 & 0 & 1 & -3 \\ -3 & 3 & -1 & 0 & 1 \\ 1 & -3 & 3 & -1 & 0 \end{pmatrix}$$

$$\simeq g(1-g)^3.$$

Mild surprise:

Four of the $S^{(2)}$'s restrict to the S's on a $\mathbb{P}^3 \subset \mathbb{P}^4$.

fractional zero-branes on $\mathbb{P}^3$.

$\phi_1 = \phi_2 \neq 0$

Where is the missing brane?

The missing brane turns out to be the zero-brane on the quintic! It lies on the point $\phi_1 + \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0$

Quantum McKay Correspondence

(Martinec -Moor

- Consider fractional zero-brane in non-supersymmetric orbifolds.

$\mathbb{C}^2/\Gamma \rightsquigarrow$ Hirzebach-Jung resolution. $\Gamma \not\subset SU(2)$

- Let $S_1, S_2, \ldots, S_r$ be the fractional branes.

- The resolution has fewer $\mathbb{P}^1$'s than the number of fractional branes, i.e., r.

The number of R's that they obtain is also less than r.

one has an imperfect matching of the R and S.

This is similar to what we saw.

One can make a somewhat more precise relation.

Recall; we considered (at 0)

$$\phi_1 + \phi_2 = 0 \;;\; \phi_3 = \phi_4 = \phi_5 = 0.$$

to get the 2-brane.

Take these four combinations, i.e.,

$$\phi_1 + \phi_2,\ \phi_3,\ \phi_4,\ \phi_5$$

to be coordinates on $\mathbb{C}^4$.

directions transverse to the two brane

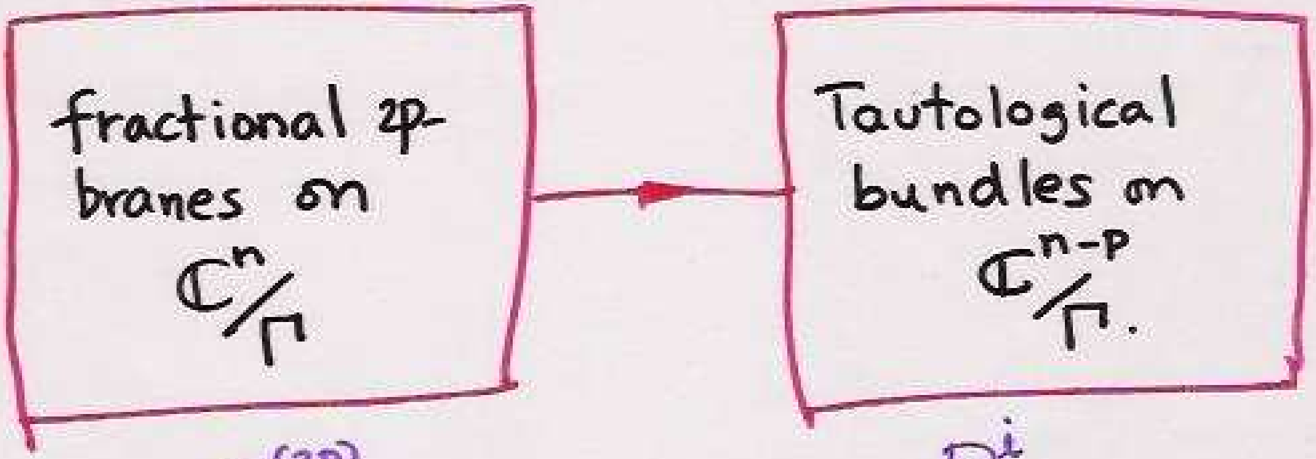
Naively, one is considering the orbifold $\mathbb{C}^4/\mathbb{Z}_5$ ← non-supersymmetry.

One then has the map.

$$\boxed{\begin{array}{c}\text{fractional}\\ \text{2-branes}\\ \text{on}\\ \mathbb{C}^5/\mathbb{Z}_5\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{fractional}\\ \text{zero-branes}\\ \text{on}\\ \mathbb{C}^4/\mathbb{Z}_5\end{array}}$$

$$S_i^{(2)} \longrightarrow S_i^{(0)} \text{ for } \mathbb{C}^4/\mathbb{Z}_5$$

Pushing the analogy further leads to the following correspondence



fractional 2p-branes on $\mathbb{C}^n/\Gamma$

Tautological bundles on $\mathbb{C}^{n-p}/\Gamma$.

$$S_i^{(2p)} \xrightarrow{\hspace{3cm}} R^i_{(2p)}.$$

perfect only when $p=0$; $\Gamma \subset SU(n)$.

A nice interpretation:

The missing branes have support away from $\mathbb{C}^{n-p} \subset \mathbb{C}^n$.

This gives a different understanding of the missing branes by the embedding of a non-supersymmetric orbifold into a supersymmetric one.

Relates to an interesting mathematical exercise.

"Champions Meet" – Reid-Craw

Conclusion and Outlook

- More formal aspects of derived categories will be discussed (?) by Koushik Ray.

- A list of topics that I've missed.

  - discussion of stability of branes [derived categories won't work].

  - how quivers and their reps. can be used to classify stable branes.
  - discussion on the superpotentials on the w.v. of branes.
  - mirror symmetry for D-branes.

- What is the CFT (Gepner model) description of the fractional two-branes? [Permutation branes?]

- Incorporating orientifolding into the story of derived categories. (SG+TJ)
  - need to use "Grothendieck duality"
  - nice relationship via quivers.
  - generalisation of Π-stability.

- The algebra whose irrep ~ spectrum of all D-branes (e.g. SO(4) for the Hydrogen atom).