BKM LIE SUPERALGEBRAS IN N = 4 SUPERSYMMETRIC STRING THEORY

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Guide : Suresh Govindarajan and S. Kalyana Rama

DECLARATION

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.
The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.
This work was done under the guidance of Prof. Suresh Govindarajan (Indian Institute of Technology Madras, Chennai) and Professor S. Kalyana Rama (Institute of Mathematical Sciences, Chennai).

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I dedicate this thesis to my parents and my sisters.

Abstract

In this thesis we study the problem of counting dyons in certain supersymmetric string theory models and the infinite dimensional Lie algebras that underlie the dyonic degeneracies. The counting of $\frac{1}{4}$-BPS states in $\mathcal{N} = 4$ supersymmetric four-dimensional string theories can be carried out in a mathematically precise and rigorous fashion due to the fact that the spectrum of these BPS states can be generated by genus-two modular forms[1, 2]. The same modular form also occurs in the context of Borcherds-Kac-Moody (BKM) Lie superalgebras[3, 4], in their denominator identities. This surprising mathematical structure underlying the spectrum of these states is the idea we develop in this thesis.

The starting point in the problem of counting dyonic states in $\mathcal{N} = 4$ supersymmetric four-dimensional string theories are two remarkable papers – one by Dijkgraaf, Verlinde and Verlinde (DVV)[1] and the other by Strominger and Vafa[5]. Strominger and Vafa provided a microscopic description of the entropy of supersymmetric black holes, which has provided enormous impetus to the counting of BPS states in a variety of settings. DVV, in a remarkable leap of intuition, proposed that the degeneracy of $\frac{1}{4}$-BPS states in the heterotic string compactified on a six torus is generated by a genus-two Siegel modular form of weight ten, $\Phi_{10}(\mathbf{Z})$.

Since then, in a series of remarkable and important papers, Sen, Jatkar and David have advanced DVV's idea to a family of asymetric orbifolds of the heterotic string compactified on T^6 leading to heterotic compactifications that preserve $\mathcal{N} = 4$ supersymmetry but with reduced gauge symmetry known as the CHL compactifications[6]. In particular, they have explicitly shown the counting of dyonic states in a special class of $\mathcal{N} = 4$ supersymmetric theories. They have also studied the dyon spectrum in $\mathcal{N} = 4$ supersymmetric type II string theories. Following this there has been enormous progress in studying and understanding the various modular forms that generate the degeneracies of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS states in these models. The modular forms in question have been generated in many different ways each being related to different aspects of the theory. DVV also observed that the modular form proposed by them as the generating function for the degeneracy of $\frac{1}{4}$-BPS states occurs as the denominator identity of a BKM Lie superalgebra studied by Gritsenko and Nikulin. This idea gives a completely new dimension to the counting of dyonic states which could not have been anticipated at the level of the action of the theory. This idea has been furthered to the models considered by Jatkar and Sen in [7, 8, 9, 10]. The BKM Lie superalgebras are related to the structure of the CHL model and it is expected that understanding the origins of this algebraic structure might provide more insight into the physics of $\frac{1}{4}$-BPS states.

The contributions of this author, along with his thesis supervisor, was to construct a new family of BKM Lie superalgebras corresponding to modular forms, $\Phi_k(\mathbf{Z})$ generating the R^2 corrections in the string effective action[7], in addition to constructing the modular forms generating the degeneracy of $\frac{1}{2}$-BPS states and $\frac{1}{4}$-BPS states for the case of non-prime N of the orbifolding group $\mathbb{Z}_N$ in the CHL strings[9]. Also, the modular forms $\widetilde{\Phi}_3(\mathbf{Z})$ and $\Phi_3(\mathbf{Z})$ as well as the BKM Lie superalgebras corresponding to the modular forms were constructed and studied. In particular, the relation between the walls of marginal stability of the $\frac{1}{4}$-BPS states and the walls of the Weyl chamber have been found to be in agreement with predictions in the literature. The connection between multiplicative η-products studied by Dummit, et. al. and the degeneracy of electrically charged $\frac{1}{2}$-BPS states has been found and the same extends to all orbifolding groups, including product groups such as $\mathbb{Z}_M \times \mathbb{Z}_N$. The author, along with collaborators has shown that the modular forms generating the degeneracy of the $\frac{1}{2}$-BPS states in the asymmetric orbifolds of the type II strings on T^6 appear as η-quotients. The modular forms that generate the degeneracy of modular forms in the type II models can be written in terms of the modular forms that appear in the CHL models. We briefly discuss the BKM Lie superalgebras in the type II models.

and studied the modular forms generating the degeneracy of the $\frac{1}{4}$-BPS states in the theory [10]. Also proposals for BKM Lie superalgebras in these models have been discussed.

List of publications/preprints

1. Suresh Govindarajan and K. Gopala Krishna. “Generalized Kac-Moody algebras from CHL dyons,” JHEP **04** (2009) 032 [arXiv:hep-th/0807.4451]

2. Suresh Govindarajan and K. Gopala Krishna, “BKM Lie superalgebras from dyon spectra in $\mathbb{Z}_N$ CHL orbifolds for composite N, "IITM/PH/TH/2009/3; IMSc/2009/04/06; [arXiv:hep-th/0907.1410]

3. Suresh Govindarajan, Dileep Jatkar and K. Gopala Krishna, “BKM Lie superalgebras from counting dyons in $\mathcal{N} = 4$ supersymmetric type II compactifications" IITM/PH/TH/2009/4; IMSc/2009/04/07 (work in progress)

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1

String Theory

1.1 Introduction

The aim of this chapter is to understand some of the basics of string theory. However, after a linear start, we will take a slightly different road map and focus on some aspects that are more natural to understand in the context of the problem that we are going to study in the following chapters. In keeping with the ‘counting’ theme of the thesis, we will look at how some of the well-known modular forms occur in connection with counting of states in string theory. The starting point of our road map is to, axiomatically, introduce a model in which the fundamental objects are strings instead of point particles. A string is a one-dimensional object – mathematically a curve. The idea, from a simple minded point of view, is to replace strings in the place of ordinary point particles and see what physics studying these leads to. We will take the theory through the same set of steps that one does to the theory of ordinary point particles we are replacing it with. To consistently do so, we should be able to recover the quantum field theory description of point particles by taking a suitable limit corresponding to length scales of everyday life. There will, however, be remnants of the stringy nature of the original theory and it will be interesting to see what the implications of these are. The original motivations for considering string theory as a theory of nature are, of course, more compelling than what is suggested here, and that by itself is a very interesting and illuminating read and we refer the reader to many of the excellent texts in string theory for it. Some suggested references are [11, 12, 13](see also [14, 15]).

There can be two fundamental kinds of strings one can consider – the open string, i.e. a string with free end points and of finite length, and the closed string, i.e. a string whose ends are joined together to form a loop, topologically equivalent to a circle. The string sweeps out a two-dimensional surface, known as the *world-sheet*, as it moves through space-time. For an open string, the topology of the world-sheet is a two-dimensional sheet, while for the case of a closed string it is a cylinder.

To make sense of it as a viable theory of nature, we also need to introduce interactions into the theory and to understand how one can generate the spectrum, containing the various elementary particles of nature, from it. Even though the basic objects of the theory are extended, the interactions in the theory need to be local in nature to preserve Lorentz covariance. Lorentz invariance of the interaction also forbids that any point on the world-sheet be singled out as the interaction point. Interactions in string theory arise when strings overlap at the same point in spacetime. The interaction results purely from the joining and splitting of strings. Consistency of the interactions force the existence of closed strings in any theory which has interacting open strings. However, one can have a consistent theory with only closed strings. The various elementary particles will be generated by the excitations of the string from its ground state. Just like the different minimal notes of a vibrational string correspond to different acoustic modes, the different vibrational modes of open and closed strings will correspond to different elementary particles. This is the model we will study in some detail below. We will first find an action for the theory, and study it. Then we quantize the theory in a suitable gauge and find its spectrum.

Let us denote the $(D+1)$-dimensional space-time manifold by $M(\sim \mathbb{R}\times M_s)$ (where M_s is the space manifold) and let $g_{\mu\nu}$ be the metric on M. The configuration of an n-dimensional object is parametrized by $(n+1)$ parameters. Thus, the two-dimensional world sheet, Σ, swept by the string, would be parametrized by two numbers $\xi^a = \{\sigma, \tau\}$, where σ (normalized to lie between $[0, l]$) is a space like coordinate and τ is timelike. Let $\gamma_{\alpha\beta}$ be the intrinsic metric on Σ. We assume that Σ and M are differentiable manifolds. The trajectory of the string in space-time is given by a set of $D+1$ functions $X^\mu(\sigma,\tau)$ which embed the world sheet Σ in the target spacetime M. The $X^\mu(\sigma,\tau)$ are continuous maps from Σ to M. $X^\mu(\sigma,\tau)$ give the position of the point (σ,τ) of the string in the space-time manifold. [1]

Like in any quantum field theory, what we want to compute from the theory are fundamental quantities like the transition amplitudes for scattering processes etc. to obtain physical predictions from the theory. Transition amplitudes in the theory have to be evaluated order by order in the loop expansion, which, in the case of a world-sheet, which is

[1] We use the Greek alphabets $\alpha, \beta, \ldots$etc. for the components of the intrinsic metric γ which take values $1, 2$, and Greek alphabets $\mu, \nu \ldots$etc. to denote space-time components of the metric g which run from $0, \ldots, D$.

a Riemannian surface, would be over varying genera. One assigns a relative weight to a given configuration and then sums over all possible configurations. In the case of strings, the configuration space is the world-sheet of the string, and the path integral is over geometries. One has to sum over all possible topologies (space of configurations) with a suitable weight to obtain the vacuum to vacuum amplitude.

To each configuration one associates the weight

$$e^{-S[X,\Sigma,M]}, \quad S \in \mathbb{C} \ ,$$

and the transition amplitude at each genus is obtained by summing over all possible metrics γ and all possible embeddings $X^\mu(\sigma,\tau)$. The functional $S[X,\gamma,M]$ is the action for the string world-sheet (we will skip the reference to the spacetime manifold from now on). The sum over all topologies is equivalent to the sum over the genera.

$$Z = \sum_\Sigma \sum_X e^{-S[X,\Sigma]} = \sum_{h=0}^{\infty} (g_s)^{2h-2} \int \mathcal{D}X\mathcal{D}\gamma e^{-S[X,\gamma]} = \sum_{h=0}^{\infty} Z_h (g_s)^{2h-2} \ .$$

Z is given as the sum of the h-loop partition functions Z_h. $\mathcal{D}\gamma$ and $\mathcal{D}X$ are the measures constructed out of diffeomorphism invariant L^2 norms on Σ and M. We need to compute them to evaluate the partition function exactly. First, however, we need to construct an action for the string to describe the motion of the string in the space-time manifold. The action should be such that all physical quantites we compute from it (like scattering amplitudes, etc.) would depend only on the embedding of the string in the space-time manifold (that is the functions $X^\mu(\sigma,\tau)$) and not on the choice of parametrization, ξ^a, of the world-sheet. Consequently, the action itself should depend only on the embedding in space-time and nothing else. It should also be consistent with the symmetries of the world-sheet and the space-time. In addition we require it to be local on its dependence on X, γ and g and be renormalizable as a QFT.

A suitable candidate is the *Nambu-Goto* action, which is proportional to the area of the world-sheet. A reformulation (and more amenable to quantization) of the Nambu-Goto action is the *Polyakov* action

$$S = -\kappa \int_\Sigma d\tau d\sigma (-\gamma)^{1/2} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \ , \tag{1.1}$$

where κ is a proportionality constant called the *string tension*. For the action to be a dimensionless quantity the string tension should have dimensions $(\text{length})^{-2} = (\text{mass})^2$ in natural units. It is taken to be $\kappa = (4\pi\alpha')^{-1}$, where α' is the Regge slope.

This, however, is not the most general action that satisfies the above mentioned criteria. There are other possible terms that one can add, like the anti-symmetric tensor field, or a dilaton field, or a tachyon field, but in this rather modestly aimed discussion we do not consider such terms.

The Polyakov action, by construction, has the symmetries of the world-sheet (the area of the world-sheet is independent of the parametrization of the world-sheet used to measure it and depends only on the embedding) and the space-time manifold (the tensor indices are properly contracted to make it Poincarë invariant) built into it. The space-time manifold is usually a pseudo-Riemannian space, which in our case, we take to be the Minkowski space-time, whose symmetries are the D-dimensional Poincaré invariance:

$$\begin{aligned} X'^{\mu}(\sigma,\tau) &= \Lambda^{\mu}_{\ \nu}\, X^{\nu}(\sigma,\tau) + a^{\mu}\ , \\ \gamma'_{\alpha\beta}(\sigma,\tau) &= \gamma_{\alpha\beta}(\sigma,\tau)\ , \end{aligned} \tag{1.2}$$

where $\Lambda \in SO(1,D)$, and $a^{\mu} \in \mathbb{R}^{D}$.

The world-sheet is a two-dimensional manifold, and has in its group of symmetries the group of diffeomorphisms $f : \Sigma_g \to \Sigma_g$ of Σ. Let $\xi^a \to \xi'^a(\xi)$ be the coordinate expression for f. The new metric is the pullback of the old one and is given by

$$\gamma_{\alpha\beta} \to f^{*}\gamma_{\alpha\beta} = \frac{\partial\xi^{\gamma}}{\partial\xi'^{\alpha}}\frac{\partial\xi^{\delta}}{\partial\xi'^{\beta}}\gamma_{\gamma\delta}\ . \tag{1.3}$$

The embedding transforms as

$$X'^{\mu}(\sigma',\tau') \to f^{*}X^{\mu} = X^{\mu}(\sigma,\tau)\ . \tag{1.4}$$

The metric $\gamma_{\alpha\beta}$ is non-dynamical in the action, and hence imposes constraints on the system. Unless we can gauge away all the independent degress of the metric, we cannot make a sensible interpretation of the physical theory. The symmetric tensor $\gamma_{\alpha\beta}$ in two dimensions has 3 independent components. The two-dimensional coordinate reparametrizations depend on two free functions and we can eliminate two of the components using this. This leaves us with one independent parameter in the metric tensor to fix. It turns out, that just for the case of two dimensions, there occurs one more local symmetry – local rescalings of the metric – that is an invariance of the classical action. It is called the *Weyl invariance* of the metric and is given by

$$\gamma'_{\alpha\beta}(\sigma,\tau) = e^{2\omega}(\sigma,\tau))\ \gamma_{\alpha\beta}(\sigma,\tau)\ , \tag{1.5}$$

for arbitrary $\omega(\sigma,\tau)$. Under a Weyl rescaling of the metric, the combination $\sqrt{\gamma}\gamma^{\alpha\beta}$ is invariant in two space-time dimensions, and thus, the action remains invariant under it. One can use this freedom to fix $\gamma_{\alpha\beta}$ (atleast locally) to be proportional to $\eta_{\alpha\beta}$. This is known as the conformal gauge. The embedding of the string in M is not affected by this change as is reflected by the transformation properties of the functions $X^\mu(\sigma,\tau)$ under the Weyl transformations

$$X'^\mu(\sigma,\tau) = X^\mu(\sigma,\tau)\ . \tag{1.6}$$

The Weyl invariance of the action, in two space-time dimensions, has very interesting and important consequences for the theory as we will discuss later.

When we quantize the theory, we will require that these symmetries be preserved if the theory is to be anomaly-free. In the conformal gauge, the action reduces to the free field action

$$S = -\kappa \int d\sigma d\tau\ \eta_{\mu\nu}\ \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu\ . \tag{1.7}$$

This choice of gauge will have to be treated more carefully when quantizing the theory. The requirement of the theory to be anomaly free will impose certain consistency conditions on the dimension of space-time and on the mass of the ground state. For now we work with the above action. Having the action, we can derive the equations of motion coming from it and find general solutions to the functions $X^\mu(\tau,\sigma)$. The Euler-Lagrange equations of motion coming from this action is just the two-dimensional linear wave equation

$$\Box X^\mu \equiv \Big(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\Big) X^\mu = 0\ . \tag{1.8}$$

This must, of course, be supplemented with the constraint equations. The constraint equations in this case are $\frac{\delta S}{\delta\gamma_{\alpha\beta}} = 0$. The variation of the action with respect to the metric γ_{ab} gives the (two-dimensional) energy-momentum tensor

$$\begin{aligned} T_{\alpha\beta}(\sigma,\tau) &= -(\kappa)^{-1}(-\gamma)^{-1/2}\frac{\delta S}{\delta\gamma^{\alpha\beta}}\ . \\ &= \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}\ \partial_\gamma X^\mu \partial^\gamma X_\mu\ . \end{aligned} \tag{1.9}$$

Thus, the constraint equation simply means that the energy-momentum tensor $T_{\alpha\beta} = 0$. The diffeomorphism invariance in two-dimensions implies the energy-momentum tensor is conserved. The Weyl invariance of the action S, which is just the statement of conformal invariance of the theory implies that the energy-momentum tensor is traceless. This means the theory is scale invariant. In two-dimensions, the conformal group is infinite-dimensional. We will get back to this remark again after we fix the boundary conditions. We will also need to examine the possibility of an anomaly in the trace of $T_{\alpha\beta}$ when quantizing the theory.

If the world-sheet has a boundary, there is also a surface term in the variation of the action. If we take the coordinate region to be

$$-\infty < \tau < \infty, \quad 0 \leq \sigma < l \ .$$

Then, the variation of the action with respect to X^μ will also have a boundary term given by

$$-\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau(-\gamma)^{1/2}\delta X^\mu \partial^\sigma X_\mu \Big|_{\sigma=0}^{\sigma=l} \ . \tag{1.10}$$

We need this term to vanish and there are different ways in which that can happen. For closed strings, one imposes a periodicity condition on the fields

$$X^\mu(\tau,l) = X^\mu(\tau,0), \quad \partial^\sigma X^\mu(\tau,l) = \partial^\sigma X^\mu(\tau,0), \quad \gamma_{\alpha\beta}(\tau,l) = \gamma_{\alpha\beta}(\tau,0) \ . \tag{1.11}$$

For the case of an open string there are two possible ways in which the boundary terms can vanish. One can require that the component of the momentum normal to the boundary of the world sheet vanish, that is,

$$\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,l) = 0 \ . \tag{1.12}$$

These are called the *Neumann* boundary conditions on the functions X^μ. The ends of the string move freely in space-time. This choice of boundary conditions means that no momentum is flowing through the ends of the string and hence it respects $D+1$-dimensional Poincaré invariance.

Alternatively one can fix the two ends of the string so that $\delta X^\mu = 0$, and

$$X^\mu\Big|_{\sigma=0} = X_0^\mu \quad \text{and} \quad X^\mu\Big|_{\sigma=l} = X_l^\mu \ , \tag{1.13}$$

where X_0^μ and X_l^μ are constants and $\mu = 0,\ldots,D$. This is known as the *Dirichlet* boundary condition. Dirichlet boundary conditions break Poincaré invariance and hence we will not consider them here. They, however, play a very important role in string theory in the study of D-branes. Once we have chosen a boundary condition, we can look for solutions to the equations of motion.

We will shift to the light-cone coordinates on the world sheet which are defined as follows

$$\sigma^{\pm} = \tau \pm \sigma \ . \tag{1.14}$$

We also define

$$\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \ . \tag{1.15}$$

With these definitions, the wave equation becomes

$$\partial_+ \partial_- X^\mu = 0 \ , \tag{1.16}$$

and the constraints involving the energy-momentum tensor become

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0 \ , \tag{1.17}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0 \ . \tag{1.18}$$

These are the *Virasoro constraints*. The conservation of the energy-momentum tensor becomes $\partial_- T_{++} + \partial_+ T_{-+} = 0$ with a similar relation for $- \leftrightarrow +$. Now, since $T_{-+} = T_{+-} = 0$ by Weyl invariance, the energy-momentum conservation equation reduces to

$$\partial_- T_{++} = 0 \ . \tag{1.19}$$

The implications of this statement are very deep. For any function $f(X^+)$ the above equation implies that the current fT_{++} is conserved as well, since $\partial_-(fT_{++}) = 0$. As f is arbitrary, this implies an infinite set of conserved quantities. These conserved quantities correspond to residual symmetries left over after we choose the covariant gauge. We will discuss more about this later.

The general solution of the wave equations is

$$X^\mu(\sigma, \tau) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \tag{1.20}$$

For a closed string satisfying periodic boundary conditions the general solution is given by

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l^2 p^\mu(\tau - \sigma) + \frac{i}{2}l \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu \exp\big(\frac{-2\pi i n(\tau-\sigma)}{\ell}\big) \ , \tag{1.21}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l^2 p^\mu(\tau + \sigma) + \frac{i}{2}l \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu \exp\big(\frac{-2\pi i n(\tau+\sigma)}{\ell}\big) \ . \tag{1.22}$$

while for an open string with Neumann boundary conditions the general solution is given by

$$X^\mu = x^\mu + l^2 p^\mu \tau + il \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma) \ , \tag{1.23}$$

where x^μ is the center-of-mass position and p^μ is the total string momentum describing the free motion of the string center of mass and the α_n^μ are Fourier components, which will be interpreted as harmonic oscillator coordinates. The parameter l is related to the Regge slope and hence the string tension κ as $l = (2\alpha')^{1/2} = (1/2\pi\kappa)^{1/2}$. The open string boundary conditions force the left and right moving modes to combine into standing waves. The right and left moving modes are independent in the closed string. The requirement that X^μ be real functions implies that α_{-n}^μ(resp. $\tilde{\alpha}_{-n}^\mu$) is the adjoint of α_n^μ(resp. $\tilde{\alpha}_n^\mu$).

We take the Fourier transform of the energy momentum tensor $T_{\alpha\beta} = 0$ at $\tau = 0$ to define the *Virasoro operators*

$$L_m = \kappa \int_0^l e^{-2im\sigma} T_{--} d\sigma \ ,$$

and

$$\tilde{L}_m = \kappa \int_0^l e^{-2im\sigma} T_{++} d\sigma \ .$$

For open strings $H = L_0$ and for closed strings $H = L_0 + \tilde{L}_0$. Clasically, the vanishing of the energy-momentum tensor translates into the vanishing of all Fourier coefficients L_m and $\tilde{L}_m$. Imposing this constraint on states leads to the mass shell condition $M^2 = -p_\mu p^\mu$ gives

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n \tag{1.24}$$

for the open string, and

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty} (\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \tag{1.25}$$

for closed strings. These determine the mass of a given string state in the quantum theory.

Before we quantize the theory, there is one more important fact to mention. The Virasoro generators L_m and $\tilde{L}_m$ satisfy an algebra amongst themselves called the Virasoro algebra given by

$$[L_m, L_n] = (m-n)L_{m+n} \ . \tag{1.26}$$

There will be a central extension to this algebra from the quantum corrections. We will not pursue this line further right now, but mention in passing that this algebra is part of a family of algebras known as *loop algebras* that we will study in some detail when we learn about infinite-dimensional Lie algebras. We now move on to quantizing the theory and find its spectrum and see what physical states it gives. We are essentially looking for representations of the Poincaré group which are unitary. We will work this out in the light-cone gauge since it is manifestly ghost free and simpler to get to the spectrum. We define the light-cone coordinates in space-time as follows.

$$X^{\pm} = (X^0 \pm X^D)/\sqrt{2} \ . \tag{1.27}$$

The light-cone gauge is obtained by setting

$$X^+(\sigma, \tau) = x^+ + p^+\tau \ .$$

In this gauge, X^- is determined by the Virasoro constraints. Thus, the only degrees of freedom are those given by direction transverse to the light-cone coordinates, $X^{\pm}$. The light-cone gauge explicitly breaks Lorentz covariance as we are singling out two of the $(D+1)$ coordinates. It should, however, be Lorentz covariant since the underlying theory it is obtained by gauge fixing from is Lorentz invariant. The conditions for the theory to preserve Lorentz invariance turn out to be identical to the constraints (we spoke of earlier) on the dimension of the space-time and the mass shell condition for the theory to be anomaly-free.

The constraint equations at the classical level require the vanishing of the components of the energy-momentum tensor. These constraints physically mean the vibrations of the embedding of the world sheet in the target space-time tangent to the surface, i.e. the longitudinal degrees of freedom, are eliminated, leaving only the $(D-1)$ transverse directions. In choosing the light-cone gauge, we are, in effect, eliminating the two longitudinal degrees of freedom and quantizing the remaining transverse degrees of freedom.

The standard way to quantize the theory is to interpret the X^μ as quantum operators and replace the Poisson brackets by commutators. The equal-time canonical commutation relations are then given by

$$[P^\mu_\tau(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta_{\mu\nu} \ ,$$
$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [P^\mu_\tau(\sigma,\tau), P^\nu_\tau(\sigma',\tau)] = 0 \ . \tag{1.28}$$

These give, for the commutation relations of the oscillator modes the following commutation relations

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] &= m\delta_{m+n}\eta^{\mu\nu} \\ [\alpha^\mu_m, \tilde{\alpha}^\mu_n] &= 0 \\ [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] &= m\delta_{m+n}\eta^{\mu\nu} \end{aligned} \tag{1.29}$$

The ground state $|0;k\rangle$, is defined to be annihilated by the lowering operators (is a highest weight state) and to be an eigenstate of the cernter-of-mass momenta,

$$P^\mu \ |0;k\rangle = k^\mu \ |0;k\rangle \ ,$$
$$\alpha^\mu_m \ |0;k\rangle = 0 \quad m > 0 \ . \tag{1.30}$$

A general state in the Fock space $\mathcal{F}_k$ can be built by acting on $|0;k\rangle$ with the raising operators.

$$|\epsilon;k\rangle = \epsilon(k, m_1, m_2, \ldots, m_n) \ \alpha^{\mu_1}_{-m_1} \cdots \alpha^{\mu_n}_{-m_n}|0;k\rangle, \tag{1.31}$$

for all possible Lorentz polarization tensors $\epsilon(k, m_1, m_2, \ldots, m_n)$, $n \in \mathbb{N}$ and all possible $m_i \in \mathbb{N}$.

The center-of-mass momenta are just the degrees of freedom of a point particle, while the oscillators represent an infinite number of internal degrees of freedom. The above equation forms the Hilbert space of a single open string. The state $|0;0\rangle$ is the ground state of a single string with zero momentum, not the zero-string vacuum state. The various operators appearing above all act within the space of states of a single string.

The open string Fock space is a sum over the Fock spaces over all momenta k. For the closed string it is a tensor product of the left and right-moving Fock spaces. A very important point to observe is that this Fock space is not positive definite. The time components have a minus sign in their commutation relations and therefore the Fock space contains states with negative norm. The physical Fock space will be a subspace of the full Fock space. We need to use the Virasoro constraints to fix an invariant subspace from the full Fock space. The Virasoro constraints in the classical theory amounted to the requirement that the components of the energy-momentum tensor, T_{++} and T_{--}, vanish. We need to impose similar conditions weakly on the quantum Fock space. The Fourier coefficients of the energy-momentum tensor were given by

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \ , \tag{1.32}$$

and a similar expression for $\tilde{L}_m$ in the case of closed strings, which we required annihilate all the physical states. In the quantum theory the α_m are operators, so one must resolve the ordering ambiguities. Since α_{m-n} commutes with α_n unless $m = 0$, we need only worry about the operator L_0. We define the L_0 operator as

$$L_0 = \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \ , \tag{1.33}$$

and define the physical state conditions with respect to L_0 and $\tilde{L}_0$ upto an undetermined constant as follows

$$\begin{aligned} |\phi\rangle \in \mathcal{F}^{phy} \quad &\text{if} \quad (L_m - a\delta_{m,0})|\phi\rangle = 0 \quad m \in \mathbb{N} \ , \\ |\phi\rangle \in \widetilde{\mathcal{F}}^{phy} \quad &\text{if} \quad (\tilde{L}_m - a\delta_{m,0})|\phi\rangle = 0 \quad m \in \mathbb{N} \ . \end{aligned} \tag{1.34}$$

The constant a is undetermined for now, and will be fixed using the condition that the physical Fock space be of positive definite norm. The mass shell condition will also undergo a modification due to the constant a as follows

$$M^2 = -2a + 2\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \ , \tag{1.35}$$

for open strings, so that the oscillator ground state has mass squared $-2a$, and excitations have mass squared larger than this by any multiple of 2. For closed strings the condition becomes (with $\alpha' = 1/2$)

$$M^2 = -8a + 8\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = -8a + 8\sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \ . \tag{1.36}$$

Imposing the condition $(L_0 - \tilde{L}_0)|\phi\rangle = 0$ we get

$$\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \ . \tag{1.37}$$

This is the only constraint equation that couples the left and right moving modes. Physical states are found by choosing independently the left-moving and right-moving states of oscillation, subject to the above constraint. The other L_m and $\tilde{L}_m$ correspond to terms of definite non zero frequency in T_{++} and T_{--}. The physical states are required to be annihilated by the positive frequency components

$$L_m|\phi\rangle = 0 \quad m = 1, 2, \ldots \tag{1.38}$$

We define the number operators(or oscillator level) as follows

$$N \equiv \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \quad , \quad \widetilde{N} \equiv \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \ . \tag{1.39}$$

They count the number of operators α_{-n} and $\tilde{\alpha}_{-n}, n \geq 1$ with weight n, applied to the ground state $|0; k\rangle$.

Writing in terms of the number operators the Virasoro constraints become

$$\begin{aligned} \text{closed} \quad & (k^2 + M^2)|\phi\rangle = 0 \quad M^2 = 8N - 8a \quad \text{ and } N = \widetilde{N}|\phi\rangle = 0 \\ \text{open} \quad & (k^2 + M^2)|\phi\rangle = 0 \quad M^2 = 2N - 2a \end{aligned} \tag{1.40}$$

It turns out (using the no-ghost theorem) that the theory is consistent if and only if the space-time dimension is 26 and the value of the constant $a = 1$. We will not prove or motivate the way this can be shown. However a fairly easy computation to check the norm of states for low mass levels shows the need for these two conditions. The bosonic string in $D = 26$ and $a = 1$ is called the *critical* bosonic string. Below we give the spectrum of the critical bosonic string.

Open String:

(i) For $N = 0$, corresponding to states of the form $|0; k\rangle$, $M^2 = -2$, hence they are *tachyons* (particles travelling faster than light), and Lorentz scalars;

(ii) For $N = 1$, corresponding to states of the form $\epsilon \cdot \alpha_{-1}|0; k\rangle$, $M^2 = 0$. These states have the degrees of freedom of a massless vector particle.

(iii) For $N = 2$ the first states with positive (mass)^2 occur. They are

$$\alpha^{\mu}_{-2}|0;k\rangle \quad \text{and} \quad \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|0;k\rangle \ , \tag{1.41}$$

with $M^2 = 2$. These states have the degrees of freedom of a massive second-rank tensor.

(iv) For higher values of N there occur various states with $M^2 > 0$ which transform under the various tensor representations of the Lorentz group.

Closed String: For closed strings there are two sets of modes corresponding to the left- and right-movers and there is the level matching condition relating the two.

(i) States corresponding to $|0;k\rangle = |0;k\rangle_L \otimes |0;k\rangle_R$ have $M^2 = -2$, so they are tachyons, and Lorentz scalars;

(ii) For $N = 1$, there occur states of the form $\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle$ have $M^2 = 0$ corresponding to the tensor product of one left-moving and one right-moving massless vector. Corresponding to the trace part of ϵ there is a Lorentz scalar, the *dilaton*, with positive norm. The symmetric traceless part of ϵ gives the *graviton*. The antisymmetric part of ϵ gives a rank two antisymmetric tensor usually denoted by $B_{\mu\nu}$.

(iii) For higher values of N there occur various states with $M^2 > 0$ which transform under the various tensor representations of the Lorentz group.

1.2 η-products from counting oscillator excitations

In this thesis, we will have occasion to consider the following trace over the open string (or left-moving sector of the closed bosonic string) physical Fock space $\mathcal{F}_k$ introduced earlier:

$$\mathrm{Tr}_{\mathcal{F}_k}\left(q^{L_0-1}\right) \ . \tag{1.42}$$

In the light-cone gauge, the full Fock space is generated by the action of all combination of the oscillator modes of the 24 transverse dimensions. Let $P_{24}(N)$ denote the number of oscillator excitations at level N arising from the 24 transverse scalars. Then, one has

$$\mathrm{Tr}_{\mathcal{F}_k}\left(q^{L_0-1}\right) = q^{E_0-1}\sum_{N=0}^{\infty} P_{24}(N)\ q^N\ , \tag{1.43}$$

where E_0 is the L_0 eigenvalue of the ground state, $|0;k\rangle$. One can show that

$$\mathrm{Tr}_{\mathcal{F}_k}\left(q^{L_0}\right) = q^{E_0}\left(\frac{1}{\prod_{n=1}^{\infty}(1-q^n)}\right)^{24} = q^{E^0}\frac{1}{\eta(\tau)^{24}}\ . \tag{1.44}$$

We thus see that modulo the ground state energy, the inverse of the product of Dedekind eta functions, $\eta(\tau)^{24}$ is the *generating function* of the oscillator degeneracy at various levels.

We will also encounter a variant of the above computation. Let g be an element of order m of a discrete group that acts on the transverse scalars and hence on their oscillator modes. The action of g on the transverse scalars can be represented by its cycle shape: $\gamma = 1^{a_1}2^{a_2}\cdots m^{a_m}$ with $\sum_{i=1}^{m} i a_i = 24$. Now consider the twisted trace

$$\mathrm{Tr}_{\mathcal{F}_k}\left(g\ q^{L_0-1}\right)\ . \tag{1.45}$$

A simple computation shows that the twisted trace (ignoring the ground state energy and related phases) is given by

$$\mathrm{Tr}_{\mathcal{F}_k}\left(g\ q^{L_0-1}\right) \sim \frac{1}{\prod_{i=1}^{m}\eta(a_i\tau)} \equiv \frac{1}{g_\gamma(\tau)}\ . \tag{1.46}$$

We see that the cycle shape γ completely determines the generating function of degeneracies of g-invariant states in the Fock space. Thus, the untwisted result corresponds to the cycle shape $\gamma = 1^{24}$. It turns out that precisely such counting problems arise in the counting of electrically charged $\frac{1}{2}$-BPS states in certain models.

1.3 Organization of The Thesis

The organization of this thesis is as follows. After a brief introduction to string theory in the introduction, in Chapter 2 we briefly review the problem we wish to study in this thesis, namely, the microscopic counting of degeneracies of BPS states in two families of string theory – the CHL models and the type II models. We discuss the counting of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS states in these theories and review the explicit counting carried out by David and Sen [16] in a class of $\mathcal{N} = 4$ supersymmetric string theories. We end the chapter with a brief review of Sen's study of the walls of marginal stability of $\frac{1}{4}$-BPS states in the CHL models.

In Chapter 3, is a self-contained review of the subject of Lie algebras. Starting with finite-dimensional semi-simple Lie algebras, we gradually introduce affine Lie algebras and finally the theory of BKM Lie superalgebras. We discuss the structure and representation theory of Lie algebras, in particular ideas like the Cartan subalgebra, root system, Weyl group, denominator formula, etc. are reviewed. We give a very brief introduction to the theory of BKM Lie superalgebras necessary to understand the denominator identity of BKM Lie superalgebras which plays a central role in this thesis. The material is presented keeping a reader with minimum mathematical background in mind.

In Chapter 4, is a self-contained introduction to the theory of modular forms. Modular forms are central to the counting problem as the generating functions of the degeneracies of the $\frac{1}{2}$- and $\frac{1}{4}$-BPS states are given by modular forms. Also, the R^2 corrections to the string effective action are given by modular forms. These modular forms are the connecting link between the string models on the one side, and the family of BKM Lie superalgebras corresponding to them, on the other.

In Chapter 5, we study the construction and properties of all the modular forms that occur in this thesis. In particular, we show how these modular forms are constructed from the additive and multiplicative lifts. We also discuss the construction of the product form of these modular forms as both the sum and product forms of the modular forms are important in understanding them as the denominator identity of BKM Lie superalgebras. We also describe the idea of cycle shapes and frame shapes that lead to genus-one modular forms generating the degeneracy of the electrically charged $\frac{1}{2}$-BPS states.

In Chapter 6, we make the connection to BKM Lie superalgebras. Given the modular forms discussed in Chapter 5, we see how they are related to a family of BKM Lie superalgebras. We review all the BKM Lie superalgebras occuring in connection with the supersymmetric string theory models considered in this thesis. We discuss the construction and properties of each of the algebras. We also discuss the relation between the walls of marginal stability discussed in Chapter 2 and the walls of the fundamental Weyl chambers of the BKM Lie superalgebras as found in [17, 8, 9].

In Chapter 7, a summary of the results obtained by the author of this thesis in work done along with collaborators is presented.

Chapter 8 concludes the thesis with an overview of the work, and future directions of research based on it.

2

Counting Dyons in String Theory

2.1 Motivation

Our motivation to undertake the microscopic counting of black hole states is an extension of our interest in understanding classical and quantum black holes in greater detail. It has been evident for quite some time now that black hole solutions of general relativity are not only physical, but also very important models in understanding quantum gravity. The Bekenstein-Hawking entropy of a black hole is one of the important aspects of black holes that can be understood both macroscopically and microscopically, thus giving us clues to understand the quantum nature of black holes. As we will see below, the counting of black hole microstates gives a way to compute the microscopic aspects of the black holes and compare it with the macroscopic side. First we will sketch the problem of counting black hole states arising from black hole thermodynamics, and then look at the spectrum from the string theory side, before finally explicitly counting black hole states in particular models.

2.2 Black Hole Thermodynamics

A black hole in the quantum theory behaves, thermodynamically, like a black body with a finite temperature, called its ***Hawking temperature***. It was shown by Hawking that such a black hole would necessarily emit radiation known now as ***Hawking radiation***. The black hole system, from a thermodynamical point of view, behaves in all respects like a blackbody with the given Hawking temperature would. In particular, it has an entropy associated to it known as the ***Bekenstein-Hawking entropy***, S_{BH}. Whenever an object falls into a black hole, the entropy carried by the object has to show up as the change in entropy of the black hole, if second law of thermodynamics is to hold. It was shown by Bekenstein that the entropy of a black hole is proportional to the area A of the event horizon as

$$S_{BH} = A/(4G_N), \tag{2.1}$$

where G_N is the Newton's constant. From a statistical point of view, however, we can understand the entropy as the logarithm of the number of microstates associated to a given macrostate at zero temperature. If Q labels the set of charges carried by a state, and $d(Q)$ the degeneracy of the states carrying this charge configuration, then the statstical entropy at *zero* Hawking temperature is given by

$$S_{stat}(Q) = \ln\, d(Q). \tag{2.2}$$

To compute S_{stat}, as defined above, would require one to have the black hole at zero Hawking temperature and also have a microscopic description of the black hole states. Unfortunately, the microscopic counting cannot be carried out for all black holes. One needs to work in special class of black holes that admit a description in terms of manageable parameters where one can exploit the symmetry structure to make the dynamics more tractable. A class of black holes known as ***extremal black holes*** have zero Hawking temperature and a high degree of tractability and so we will be looking at black hole solutions that are extremal. We will now motivate such a model. For a general introduction to black hole thermodynamics see [18, 19]

2.2.1 The Reissner-Nordström Black Hole

We start with the following $3+1$-dimensional action in the Einstein-Maxwell theory with terms upto two derivatives

$$S = \int d^4\sqrt{-g}\Big[\frac{1}{16\pi G_N}R - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big]\ . \tag{2.3}$$

We look for static solutions having spherical symmetry. For a non-charged black hole this leads to the Schwarzschild black hole solution. Looking for solutions which have electric and magnetic charges leads to the Reissner-Nordström black hole solution

$$ds^2 = -f(\rho)d\tau^2 + f^{-1}(\rho)d\rho^2 + \rho^2 d\Omega^2,$$

$$F_{\rho\tau} = \frac{q_e}{4\pi\rho^2}, \quad F_{\theta\phi} = \frac{q_m}{4\pi}\sin\theta, \tag{2.4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the two-sphere, and $f(\rho)$ is given in terms of the the ADM mass and the charges (q_m, q_e) by

$$f(\rho) = 1 - \frac{2G_N M}{\rho} + \frac{G_N}{4\pi\rho^2}(q_e^2 + q_m^2) \ . \tag{2.5}$$

One can recognize the Schwarzschild black hole in the above solution when $q_e = q_m = 0$. The solution (2.4) has a singularity at $r = 0$. The black hole would be stable at the extremal limit corresponding to choosing $M^2 = \frac{1}{4\pi G_N}(q_e^2 + q_m^2)$. In the extremal limit, the Hawking temperature of the black hole is zero, and hence the black hole no longer radiates. If it radiated, M^2 would become less than $\frac{1}{4\pi G_N}(q_e^2 + q_m^2)$ and the condition would no longer hold and produces a naked singularity. Thus, the black hole solution in the extremal limit characterizes a black hole which is the stable endpoint of Hawking evaporation. The entropy of the black hole remains finite and is given by

$$S_{BH} = \tfrac{1}{4}(q_e^2 + q_m^2) \ . \tag{2.6}$$

Following Sen [20] we define

$$t = \lambda\tau/a^2, \quad r = \lambda^{-1}(\rho - a), \tag{2.7}$$

where,

$$a = \sqrt{\frac{G_N}{4\pi}(q_e^2 + q_m^2)} \ . \tag{2.8}$$

and λ is an arbitrary constant, and taking the 'near horizon' limit, $\lambda \to 0$, the solution (2.4) becomes

$$\begin{aligned} ds^2 &= a^2\Big(-r^2dt^2 + \frac{dr^2}{r^2}\Big) + a^2(d\theta^2 + \sin^2\theta d\phi^2), \\ F_{r,t} &= \frac{q_e}{4\pi}, \quad F_{\theta\phi} = \frac{q_m}{4\pi}\sin\theta \ . \end{aligned} \tag{2.9}$$

which is a product of two spaces. The space labelled by (θ, ϕ) is the two-dimensional sphere S^2. The space labelled by (r, t) is the two-dimensional AdS_2 space-time. The AdS_2 space-time is a solution of two-dimensional Einstein gravity with a negative cosmological constant. The spherical symmetry of the black hole solution manifests as an $SO(3)$ isometry acting on the S^2. In addition there is also an $SO(2,1)$ isometry acting on the AdS_2 that was not present in the full black hole solution. All known extremal spherically symmetric black hole solutions in four-dimensions with non-singular horizon have near horizon geometry $AdS_2 \times S^2$ and an associated $SO(2,1) \times SO(3)$ isometry. Now, when we consider the action beyond the two derivative terms, ***we will postulate that the higher derivative terms we add in the action would not destroy the near horizon symmetries***[20]. For black holes with large curvature at the horizon the higher derivative terms are as important as the two derivative terms we have considered here. Thus, we will assume that in any generally covariant theory of gravity coupled to matter fields, the near horizon geometry of a spherically symmetric extremal black hole in four-dimensions has the above mentioned $SO(2,1) \times SO(3)$ isometry. Following Sen[20], we shall take this as the definition of a spherically symmetric extremal black hole in four-dimensions.

2.3 Black Holes in String Theory

We want to investigate the above ideas in the context of string theory. We look for solutions which are static and have the $SO(2,1) \times SO(3)$ isometry. Even with these symmetry constraints the solutions are fairly complicated with the scalar fields depending non-trivially on the radial direction and we need to find an analogue of the conditions that lead to extremality to be able to find tractable solutions. The first smooth solutions were constructed compactifying the heterotic string on T^6[21, 22, 23]. Charged solutions have the same structure as the Reissner-Nordström solution. Often these black holes are also invariant under certain number of supersymmetry transformations and in that case they are known as BPS black holes, and the analogue of the conditions that led to extremality for the Reissner-Nordström solution in this context is the necessity of the saturation of the BPS bound[1]. The saturation of the bound implies that the black hole preserves some fraction of the supersymmetry of the vacuum. We can thus obtain black hole solutions with the two important properties – stability and symmetry – which make the extremal Reissner-Nordström solution tractable. One can calculate the degeneracy of such states at weak coupling and hence the entropy, at weak coupling. Supersymmetry ensures that we can continue the result to the strong coupling regime where the system can be best described as a black hole.

Another reason extremal black holes are particularly suitable to work with is the so called "attractor mechanism" as a result of which, the entropy is independent of asymptotic

[1]While it is not always true that the BPS bound coincides with the extremal limit, it will be true in all our considerations.

values of the moduli scalar fields[24, 25, 26]. Thus the entropy of an extremal black hole does not change as we change the asymptotic values of the string coupling constant from a sufficiently large value where it has a black hole description to a much smaller value where the microscopic description is valid.

There are, however, two sides to the symmetry of the system. On the one side the high degree of symmetry of the theory lets one deduce many features of the theory using symmetry arguments alone, and hence gives a way of understanding the system. On the other hand, since our final aim is understanding general systems with no symmetry, studying systems with such high degree of symmetry can help us only so much. However, as a first step it is educative to understand extremal solutions to test the validity of the procedure by computing and comparing quantities of the black hole system that can be computed from other methods. One such important computation is of the entropy of extremal black holes which can be computed from macrocsopic parameters as in eq. (2.1) and comparing it with a microscopic counting of the states of the black hole.

It, however, took a while before such explicit computations could be realized. In 1995, Strominger and Vafa pioneered the idea of thinking of the black hole as a bound state of solitons (D-branes) in string theory, and using the stability of the BPS states to continue the solution to the weak coupling limit[5]. Since then many similar computations have been carried out for the case of extremal and near-extremal black holes[27, 28, 29]. In the limit where the size of the black hole is large, the Bekenstein-Hawking entropy S_{BH} has been found to be the same as the statistical entropy of the same charge configuration. i.e.

$$S_{BH}(Q) = S_{\rm stat}(Q). \tag{2.10}$$

The above comparisons between S_{BH} and S_{stat} were initially carried out in the large charge limit, where the horizon size is large so that the curvature and other field strengths at the horizon are small and hence we can ignore them.

Typically string theory compactified to four-dimensions involves many more fields than appearing in the Einstein-Maxwell action we considered above. Requiring $\mathcal{N} \geq 2$ supersymmetry in the solutions generically gives theories with abelian gauge fields, massless scalars and their fermionic partners. We will primarily be interested in studying solutions in four-dimensional space-time with $\mathcal{N} = 4$ supersymmetry. In particular, we will focus on two classes of four-dimensional compactifications:

(i) Asymmetric orbifolds of the heterotic string on T^6 – *the CHL models*,

(ii) Asymmetric orbifolds of the type IIA string on T^6 – *the type II models.*

Before describing these models in greater detail we will discuss a proposal of Dijkgraaf, Verlinde, and Verlinde (DVV) of representing the generating function of degeneracies of dyonic states by automorphic forms and its compatibility with the macroscopic entropy of dyonic black holes[1]. This is the genesis of subsequent proposals by David, Jatkar and Sen for representing dyonic degeneracies in terms of automorphic forms for the two classes of models[30, 31, 32, 2, 33, 34, 35, 36].

2.4 The DVV Proposal

DVV considered the heterotic string compactified on a six-torus. In this four-dimensional theory (which is dual to type II theory on $K3 \times T^2$), dyonic states carry 28 electric and 28 magnetic charges, denoted $\mathbf{q}_e$ and $\mathbf{q}_m$, respectively, living on an even self dual lattice $\Gamma^{22,6}$. This theory has as its duality group $SL(2,\mathbb{Z}) \times SO(22,6,\mathbb{Z})$, where the $SL(2,\mathbb{Z})$ is the electric-magnetic duality symmetry. The purely electric states, which arise perturbatively as heterotic string states, can be counted easily since they preserve half of the supercharges and hence simply correspond to the heterotic string states in the right-moving ground state[2]. Hence the number of such states can be computed if one specifies the 28 electric charges along with their occupation numbers, subject to the level matching condition

$$\tfrac{1}{2}\mathbf{q}_e^2 + \sum_{\ell,I} \ell N_\ell^I = 1, \tag{2.11}$$

where the subscript ℓ denotes the world-sheet oscillator number of the coordinate field x^I, and the scalar product on the lattice $\Gamma^{22,6}$ is defined using the $SO(22,6,\mathbb{Z})$ invariant inner product. The number of such states is

$$d(\mathbf{q}_e) = \oint d\sigma \frac{e^{i\pi\sigma\mathbf{q}_e^2}}{\eta(\sigma)^{24}}, \tag{2.12}$$

where the contour integral over σ is from 0 to 1 and $\eta(\sigma)$ is the Dedekind η-function.

The magnetic charges do not arise perturbatively but as solitonic states. From the electric-magnetic duality, there should also exist a solitonic version of the heterotic string that carries pure magnetic charge $\mathbf{q}_m \in \Gamma^{22,6}$, and hence a similar formula that counts the

[2]We choose the right-movers to be supersymmetric for the heterotic string.

magnetic charges. Thus, we have a generating function for the degeneracies of the $\frac{1}{2}$-BPS states given in terms of the Dedekind eta products.

The generic dyonic states, however, preserve only one-quarter of the supersymmetries and hence will be a bigger set whose degeneracies will be given by a more general formula. The formula will be a generalization of (2.12) and should reduce to it when the supersymmetry is restored back to half. DVV proposed a formula for the degeneracies of the $\frac{1}{4}$-BPS states on the idea that the $\frac{1}{4}$-BPS states are a bound state of an electric heterotic string with a dual magnetic heterotic string. It is actually an index in that it counts the number of bosonic minus the fermionic BPS-multiplets for a given configuration of electric and magnetic charge.

2.4.1 The Degeneracy Formula

For convenience, we combine the electric and magnetic charge vectors into a single vector as

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_e \\ \mathbf{q}_m \end{pmatrix},$$

and introduce the matrix

$$\mathbf{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}, \tag{2.13}$$

generalizing the single modulus σ in (2.12). DVV proposed that the degeneracy of the $\frac{1}{4}$-BPS states is then given by

$$64 d(\mathbf{q}_e, \mathbf{q}_m) = \oint d\mathbf{Z}\, \frac{e^{i\pi \mathbf{q}^{\mathrm{T}} \cdot \mathbf{Z} \cdot \mathbf{q}}}{\Phi_{10}(\mathbf{Z})}\,. \tag{2.14}$$

The integrals over the moduli parameters σ, ρ and ν run over the domain from 0 to 1 and we need to impose the level matching condition as before. The matrix $\mathbf{Z}$ is the period matrix of a genus-two Riemann surface and the function $\Phi_{10}(\mathbf{Z})$ is genus-two modular form which is the unique automorphic form of weight 10 of the modular group $Sp(2,\mathbb{Z})$. The $SL(2,\mathbb{Z})$ duality transformations are identified with the subgroup of $Sp(2,\mathbb{Z})$ that leave the genus-two modular form $\Phi_{10}(\mathbf{Z})$ invariant. Thus, the degeneracy formula is manifestly duality symmetric. $\Phi_{10}(\mathbf{Z})$ also has a representation in terms of the product of genus-2 theta constants as [3]

$$\Phi_{10}(\mathbf{Z}) = \Big(\frac{1}{64}\prod_{m=0}^{9} \theta_m(\mathbf{Z})\Big)^2\,. \tag{2.15}$$

[3]The expression of Siegel modular forms as products of even genus-two theta constants is discussed in chapter 5.

Another equivalent representation of $\Phi_{10}(\mathbf{Z})$ is obtained as an infinite product representation from the Fourier coefficients of the elliptic genus of $K3$

$$\chi_{K3}(\tau, z) = \mathrm{Tr}(-1)^{F_L+F_R} e^{2\pi i(\tau(L_0-\frac{c}{24})+zF_L)}, \tag{2.16}$$

with $c = 6$ for $K3$ and F_L and F_R are the space-time fermion numbers which can be identified with the zero-modes of the left-moving and right-moving $U(1) \subset SU(2)$ current algebras. The automorphic form $\Phi_{10}(\mathbf{Z})$ is given as an infinite product by

$$\Phi_{10}(\rho, \sigma, \nu) = e^{2\pi i(\rho+\sigma+\nu)} \prod_{(k,l,m)>0} \left(1 - e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{c(4kl-m^2)}, \tag{2.17}$$

where $(k, l, m) > 0$ means that $k, l \geq 0$ and $m \in \mathbb{Z}$, $m < 0$ for $k = l = 0$, and the coefficients $c(n)$ are defined by the expansion of the $K3$ elliptic genus as

$$\chi_{K3}(\tau, z) = \sum_{h\geq 0, m\in\mathbb{Z}} c(4h - m^2)\, e^{2\pi i(h\tau+mz)}, \tag{2.18}$$

where, up to normalization, $\chi_{K3}(\tau, z)$ is the unique weak Jacobi form of index 1 and weight 0. Note that $c(n) = 0$ for $n < -1$.

We will conclude our study of the DVV proposal with a few remarks on checking its consistency. We will take the Fourier transform of (2.14) and write it as

$$\frac{64}{\Phi_{10}(\mathbf{Z})} = \sum_{k,\ell,m} D(k, \ell, m) e^{-2\pi i(k\rho+\ell\sigma+m\nu)}, \tag{2.19}$$

with $k, \ell, m \in \mathbb{Z}$. The coefficients $D(k, \ell, m)$ are all integers and are related to the degeneracies (2.14) by

$$d(\mathbf{q}_e, \mathbf{q}_m) = D(\tfrac{1}{2}\mathbf{q}_e^2, \tfrac{1}{2}\mathbf{q}_m^2, \mathbf{q}_e \cdot \mathbf{q}_m)\ . \tag{2.20}$$

Now, as a first consistency check, we want to see if one can obtain (2.12) as a limit of (2.14). The parameter ν couples to the helicity m of the dyonic states, and thus the integral over ν projects out dyons with helicity equal to zero. However, instead of integrating it out, one can also put it equal to a fixed value, like $\nu = 0$. In the $\nu = 0$ limit then, we will obtain the formula with a helicity trace $(-1)^m$, which will project out the $\frac{1}{4}$-BPS states and will leave only the $\frac{1}{2}$-BPS states whose degeneracy is given by (2.12). Taking the $\nu \to 0$ limit in (2.19) we get

$$\lim_{\nu\to 0} \frac{e^{i\pi \mathbf{q}\cdot\mathbf{Z}\cdot\mathbf{q}}}{\Phi_{10}(\mathbf{Z})} \longrightarrow \frac{1}{\nu^2}\,\frac{e^{i\pi\rho \mathbf{q}_e^2}}{\eta(\rho)^{24}}\,\frac{e^{i\pi\sigma \mathbf{q}_m^2}}{\eta(\sigma)^{24}}\ . \tag{2.21}$$

The above formula shows that the genus-two surface parametrized by $\mathbf{Z}$ factors into two separate genus-one surfaces with moduli ρ and σ which correspond to the $\frac{1}{2}$-BPS moduli. As another non-trivial check for the degeneracy formula, we can compare the macroscopic Bekenstein-Hawking entropy of extremal four-dimensional black holes with the asymptotic behaviour for large charges of (2.14). The degeneracy formula (2.14) matches the macroscopic entropy results in the large charge limit.

With this brief introduction into the DVV proposal, we now turn to the two models that we will study for the rest of this thesis – the CHL and type II models. We will derive a formula similar to the above one for the degeneracies of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS states for the case of CHL strings, and following David, Jatkar and Sen show how the modular form in question is generated by an explicit counting of the black hole microstates in the model. We will also look at the modular forms generating the dyonic degeneracies in the type II models. We first start by describing, briefly, the CHL and type II models below, before moving on to understanding the explicit counting of the dyonic degeneracies in a $D = 4$, $\mathcal{N} = 4$ supersymmetric model.

2.5 The CHL Models

The heterotic string compactified on T^6 and its asymmetric $\mathbb{Z}_N$ orbifolds provide us with four-dimensional compactifications with $\mathcal{N} = 4$ supersymmetry. Writing T^6 as $T^4 \times \widehat{S}^1 \times S^1$, consider the $\mathbb{Z}_N$ orbifold given by the transformation corresponding to a $1/N$ unit of shift in $\widehat{S}^1$ and a simultaneous $\mathbb{Z}_N$ involution of the Narain lattice of signature $(4, 20)$ associated with the heterotic string compactified on T^4. This leads to the CHL models that we will study[2]. Starting from six-dimensional string-string duality, one sees that the heterotic string compactified on $T^4\times\widetilde{S}^1\times S^1$ is dual to the type IIA string compactified on $K3\times\widetilde{S}^1\times S^1$. The $(4, 20)$ lattice gets mapped to $H^*(K3, \mathbb{Z})$ in the type IIA theory and the orbifolding $\mathbb{Z}_N$ is a Nikulin involution combined with the $1/N$ shift of $\widetilde{S}^1$. There is a third description that is obtained by T-dualizing the $\widehat{S}^1$ to $\widetilde{S}^1$ and following it by an S-duality – this is used to carry out the microscopic counting. Figure 2.1 summarizes the chain of dualities.
The low-energy theory consists of the following bosonic fields:

(i) the $\mathcal{N} = 4$ supergravity multiplet with the graviton, a complex scalar, S_H and six

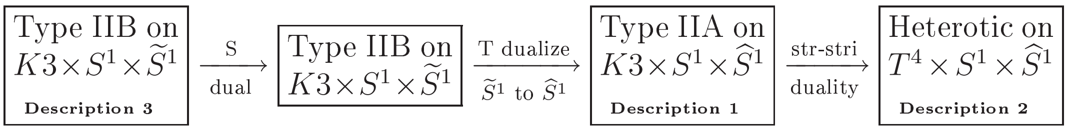

Figure 2.1: The chain of dualities in the CHL model. The above chain is expected to hold after $\mathbb{Z}_N$-orbifolding of $K3 \times S^1$. The quantization of charges is specified in Description 2 (asymmetric orbifolds of the heterotic string i.e., CHL strings) while microscopic counting is carried out in Description 3.

graviphotons; and

(ii) m $\mathcal{N}=4$ vector multiplets each containing a gauge field and six scalars[4].

In terms of the variables that appear in the heterotic description, the bosonic part of the low-energy effective action (up to two derivatives) is[37, 21, 38]

$$S = \int d^4x\sqrt{-g}\left[R - \frac{\partial_\mu S_H\ \partial^\mu \bar{S}_H}{2\ \mathrm{Im}(S_H)^2} + \frac{1}{8}\mathrm{Tr}(\partial_\mu ML\ \partial^\mu ML)\right. \\ \left. -\frac{1}{4}\mathrm{Im}(S_H)\ F_{\mu\nu}LML\ F^{\mu\nu} + \frac{1}{4}\mathrm{Re}(S_H)\ F_{\mu\nu}L\ \widetilde{F}^{\mu\nu}\right]\ , \quad (2.22)$$

where L is a Lorentzian metric with signature $(6,m)$, M is a $(6+m)\times(6+m)$ matrix valued scalar field satisfying $M^T = M$ and $M^TLM = L$ and $F_{\mu\nu}$ is a $(6+m)$-dimensional vector field strength of the $(6+m)$ abelian gauge fields. The moduli space of the scalars is given by

$$\big(\Gamma_1(N)\times SO(6,m;\mathbb{Z})\big)\Big\backslash\left(\frac{SL(2)}{U(1)}\times\frac{SO(6,m)}{SO(6)\times SO(m)}\right) \quad (2.23)$$

$SO(6,m;\mathbb{Z})$ is the T-duality symmetry and $\Gamma_1(N)\subset SL(2,\mathbb{Z})$ is the S-duality symmetry that is manifest in the equations of motion and is compatible with the charge quantization[39]. The fields that appear at low-energy can be organized into multiplets of these various symmetries.

1. The heterotic dilaton combines with the axion (obtained by dualizing the antisymmetric tensor) to form the complex scalar S_H. Under S-duality, $S_H \to (aS_H+b)/(cS_H+d)$.

2. The $(6+m)$ vector fields transform as a $SO(6,m;\mathbb{Z})$ vector. Thus, the electric charges $\mathbf{q}_e$ (resp. magnetic charges $\mathbf{q}_m$) associated with these vector fields are also vectors (resp.

[4]When $N=1,2,3,5,7$, $m=([48/(N+1)]-2)$

co-vectors) of $SO(6, m, \mathbb{Z})$. Further, the electric and magnetic charges transform as a doublet under the S-duality group, $\Gamma_1(N)$, where $\Gamma_1(N)$ is the sub-group of $SL(2,\mathbb{Z})$ matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a = d = 1 \bmod N$ and $c = 0 \bmod N$).

One can form three T-duality invariant scalars, $\mathbf{q}_e^2$, $\mathbf{q}_m^2$ and $\mathbf{q}_e \cdot \mathbf{q}_m$ from the charge vectors. These transform as a triplet of the S-duality group. Equivalently, we can write the triplet as a symmetric matrix:

$$\mathcal{Q} \equiv \begin{pmatrix} \mathbf{q}_e^2 & -\mathbf{q}_e \cdot \mathbf{q}_m \\ -\mathbf{q}_e \cdot \mathbf{q}_m & \mathbf{q}_m^2 \end{pmatrix} \tag{2.24}$$

The S-duality transformation now is $\mathcal{Q} \to A \cdot \mathcal{Q} \cdot A^T$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$. The charges are quantized such that $N\mathbf{q}_e^2,\ \mathbf{q}_m^2 \in 2\mathbb{Z}$ and $\mathbf{q}_e \cdot \mathbf{q}_m \in \mathbb{Z}$. There exist many more invariants due to the discrete nature of the T-duality group[40] for $N = 1$ and more appear when $N > 1$.

2.5.1 BPS Multiplets

Four-dimensional compactifications with $\mathcal{N} = 4$ supersymmetry admit two kinds of BPS states: (i) $\frac{1}{2}$-BPS multiplets that preserve eight supercharges (with 16 states in a multiplet) and (ii) $\frac{1}{4}$-BPS multiplets that preserve four supercharges(with 64 states in a multiplet). The masses of the $\frac{1}{4}$-BPS states are determined in terms of their charges by means of the BPS formula[21, 41, 38]:

$$\begin{aligned}\left(M_{\pm}^2\right)_{\frac{1}{4}-\text{BPS}} = \frac{1}{S_H - \bar{S}_H} \Big[&(\mathbf{q}_e + S_H \mathbf{q}_m)^T (M+L)(\mathbf{q}_e + \bar{S}_H \mathbf{q}_m) \\ &\pm \frac{1}{2} \sqrt{(\mathbf{q}_e^T (M+L)\mathbf{q}_e)(\mathbf{q}_m^T (M+L)\mathbf{q}_m) - (\mathbf{q}_e^T (M+L)\mathbf{q}_m)^2}\ \Big] .\end{aligned} \tag{2.25}$$

The square of the mass of a $\frac{1}{4}$-BPS state is $\max(M_+^2, M_-^2)$. 1/2-BPS states appear when the electric and magnetic charges are parallel (or anti-parallel) i.e., $\mathbf{q}_e \propto \mathbf{q}_m$. The BPS mass formula for $\frac{1}{2}$-BPS states can be obtained as a specialization of the $\frac{1}{4}$-BPS mass formula given above. When $\mathbf{q}_e \propto \mathbf{q}_m$, the terms inside the square root appearing in the $\frac{1}{4}$-BPS mass formula vanish leading to the $\frac{1}{2}$-BPS formula

$$\left(M^2\right)_{\frac{1}{2}-\text{BPS}} = \frac{1}{S_H - \bar{S}_H} \Big[(\mathbf{q}_e + S_H \mathbf{q}_m)^T (M+L)(\mathbf{q}_e + \bar{S}_H \mathbf{q}_m) \Big] . \tag{2.26}$$

2.5.2 Counting $\frac{1}{2}$-BPS States

We will now consider the counting of purely electrically charged $\frac{1}{2}$-BPS states. Such electrically charged states are in one to one correspondence with the states of the CHL orbifold of the heterotic string compactified on $T^4 \times S^1 \times \widehat{S}^1$[39]. Let $d(n)$ denote the degeneracy of heterotic string states carrying charge $N\mathbf{q}_e^2 = 2n$ – the fractional charges arise from the twisted sectors in the CHL orbifolding. Every $\frac{1}{2}$-BPS multiplet/heterotic string state has degeneracy $16 = 2^{8/2}$. Then the generating function of $d(n)$ is[32, 42, 43] (for $N = 1, 2, 3, 5, 7$ and $k + 2 = 24/(N + 1)$)

$$\sum_{n=0}^{\infty} d(n)\ \exp(\tfrac{2\pi i n\tau}{N}) = \frac{16}{(i\sqrt{N})^{-k-2}\ f^{(k)}(\tau/N)}\ , \tag{2.27}$$

where

$$f^{(k)}(\tau) \equiv \eta(N\tau)^{k+2}\ \eta(\tau)^{k+2}\ . \tag{2.28}$$

The degeneracy of purely magnetically charged states with charge $\mathbf{q}_m = 2m$ is given by a similar formula with $f^{(k)}(\tau/N)$ replaced by $f^{(k)}(\tau)$. These are level-N genus-one modular forms with weight $(k + 2)$. For $(N, k) = (1, 10)$, $f^{(10)}(\tau) = \eta(\tau)^{24}$.

2.5.3 Counting $\frac{1}{4}$-BPS States

As we saw earlier, $\frac{1}{4}$-BPS states are necessarily dyonic in character with the electric and magnetic charge vectors being linearly independent. Jatkar and Sen generalized the DVV proposal to the case of asymmetric $\mathbb{Z}_N$-orbifolds of the heterotic string on T^6 for $N = 2, 3, 5, 7$[2]. They proposed that the egeneracy of $\frac{1}{4}$-BPS dyons is generated by a Siegel modular form of weight $k = \frac{24}{N+1} - 2$ and level N, $\widetilde{\Phi}_k(\mathbf{Z})$. They also provided an explicit construction of the modular form using the additive lift of a weak Jacobi form. The constructed modular form has the following properties:

(i) It is invariant under the S-duality group $\Gamma_1(N)$ suitably embedded in the group $G_1(N) \subset Sp(2, \mathbb{Z})$. (See Appendix **D** for notation)

(ii) In the limit $z_2 \to 0$, it has the right factorization property:

$$\lim_{z_2\to 0}\ \widetilde{\Phi}_k(\mathbf{Z}) = (i\sqrt{N})^{-k-2}\ (2\pi z_2)^2\ f^{(k)}(z_1/N)\ f^{(k)}(z_3) \tag{2.29}$$

Note that for $(N, k) = (1, 10)$, this matches the DVV formula (eq. (2.21)).

(iii) It reproduces the entropy for large blackholes[2].

(iv) For $\frac{1}{2}$-BPS blackholes, the formula leads to a prediction for R^2 (higher derivative) corrections to the low-energy effective action given in Eq. (2.22). Such corrections lead to a non-zero entropy using Wald's generalization of the BH entropy formula for Einstein gravity that agrees with the prediction from the modular form[44, 45].

With this brief introduction, we move on to the other model we will consider in this work – the type II compactifications with $\mathcal{N} = 4$ supersymmetry. These models are similar to the CHL models with the $K3$, appearing in the type IIA/B description, being replaced by T^4.

2.6 The Type II Models

Type II string theory compactified on a six-torus has $\mathcal{N} = 8$ supersymmetry in four-dimensions. We will consider fixed-point free $\mathbb{Z}_N$ ($N = 1, 2, 3, 4, 5$) orbifolds of the six-torus that preserve $\mathcal{N} = 4$ supersymmetry. The orbifold procedure involves splitting $T^6 = T^4 \times S^1 \times \widetilde{S}^1$ and choosing the action of $\mathbb{Z}_N$ such that it has fixed points on T^4, but this action is accompanied by a simultaneous $1/N$ shift along the circle S^1. The total action of the orbifold is free, *i.e.*, it has no fixed points. It thus suffices to specify the action of T^4.

As we will be moving between several descriptions of the orbifold related by duality, we will need to specify the duality frame. *Description one* corresponds to type IIA string theory on a six-torus with the following $\mathbb{Z}_N$ action.

$N \neq 5$ Let $\omega = \exp(2\pi i/N)$ and (z_1, z_2) be complex coordinates on T^4. The $\mathbb{Z}_N$ action is generated by $(z_1, z_2) \to (\omega z_1, \omega^{-1} z_2)$.

$N = 5$ Let $\omega = \exp(2\pi i/5)$ and $T^4 = \mathbb{R}^4/\Gamma_{A_4}$, where Γ_{A_4} is the root lattice of Lie algebra A_4. The $\mathbb{Z}_5$ generator has eigenvalues ω^r with $r = 1, 2, 3, 4 \bmod 5$. This corresponds to a quasi-crystalline compactification.

Our considerations generalize the $N = 2, 3$ orbifolds considered in [31]. Again a chain of dualities (discussed later in this chapter) relates this to other type IIA/B compactifications. In particular, the analog of the CHL string turns out to be the type IIA string – see Figure 2.2.

The two-derivative low-energy effective action is constrained by supersymmetry and the number of vector multiplets and is identical to the one discussed for the CHL model (see eq.

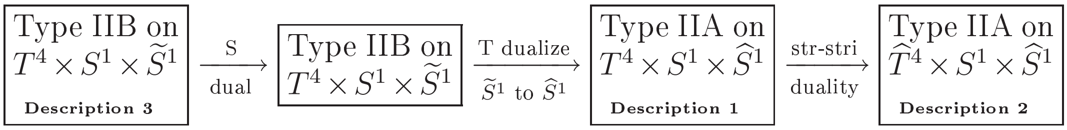

Figure 2.2: The chain of dualities in the type II models. The above chain is expected to hold after $\mathbb{Z}_N$-orbifolding of $T^4 \times S^1$. The quantization of charges is specified in Description 2 (asymmetric orbifold of the type IIA string) while microscopic counting is carried out in Description 3.

(2.22)). Similarly, the mass formulae for $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS states given in the CHL model also hold here with S_H being identified with the dilaton in description 2.

2.6.1 $\mathbb{Z}_N$-Action From the NS5-Brane

Under six-dimensional string-string duality, type IIA string on T^4 (description 1) is dual to type IIA string on the T-dual torus $\widehat{T}^4$ (description 2). The dual type IIA string is a soliton obtained by wrapping the NS5-brane on T^4. We are interested in the situation where this is compactified to four-dimensions and there is a $\mathbb{Z}_N$ orbifold action as mentioned above. Vafa and Sen have obtained the corresponding orbifold action (for $N = 2, 3$) in the dual description[46]. We will obtain their result and its generalization for the $N = 4, 5$ orbifolds by studying the $\mathbb{Z}_N$ action in effective 1+1-dimensional worldvolume theory of the NS5-brane on T^4 (see [47, 48] for a related discussion).

The fields in the worldvolume theory of a single NS5-brane consist of five scalars, a second-rank antisymmetric tensor (with self-dual field strength) in the bosonic sector and four chiral fermions. These are the components of a single $(2, 0)$ tensor multiplet in $5 + 1$-dimensions. We can dimensionally reduce the fields on T^4 to obtain the fields on an effective $1 + 1$-dimensional theory. Using string-string duality, this theory will be that of a type IIA Green-Schwarz string in the light-cone gauge [47, 48].

Let us organize the fields in terms of $SO(4) \times SO(4)_R$ where the first $SO(4) = SU(2)_L \times SU(2)_R$ is from the T^4 and the R-symmetry can be taken to be rotations about the four transverse directions to the NS5-brane.

1. Four scalars, x^m, are in the representation $(1, 4_v)$. These become four non-chiral scalars on dimensional reduction on the four-torus.

2. The fifth scalar and the two-form antisymmetric gauge field can be combined and written as $Y_{\alpha\beta}$ and $Y_{\dot{\alpha}\dot{\beta}}$ where α is a $SU(2)_L$ spin-half index and $\dot{\beta}$ is a $SU(2)_R$ spin-

half index. On dimensional reduction on the four-torus, the $Y_{\alpha\beta}$ become the four left-moving chiral bosons and the $Y_{\dot{\alpha}\dot{\beta}}$ become four right-moving chiral bosons. When combined with the four non-chiral bosons, they become the Green-Schwarz bosons in the light-cone gauge of the type IIA string.

3. The fermions are $\psi_{A\beta}$ and $\psi_{A\dot{\beta}}$ where A is a spinor index of $SO(4)_R$. These become the left- and right-moving fermions in the effective $1+1$-dimensional theory — these are the Green-Schwarz fermions in the light-cone gauge of the type IIA string.

In the above set up, the transformations under that the $\mathbb{Z}_N$ subgroup of $SU(2)_L$ is given by

$$g_{\alpha}{}^{\beta} \equiv \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \ , \tag{2.30}$$

where $\omega = \exp(2\pi i/N)$ for $N = 2, 3, 4$.

One can see that the only fields that transform under this action are those that carry the index α. Thus, we see that the chiral fermions all transform as

$$\psi_{A\alpha} \to g_{\alpha}{}^{\beta}\ \psi_{A\beta} \ . \tag{2.31}$$

Thus we see that 4 of the fermions pick up the phase ω and the other four pick up the phase ω^{-1}. The field $Y_{\alpha\beta}$ transforms as

$$Y_{\alpha\beta} \to g_{\alpha}{}^{\gamma}\ g_{\alpha}{}^{\delta}\ Y_{\gamma\delta} \ . \tag{2.32}$$

Thus, two fields are invariant under the $\mathbb{Z}_N$ and the other two transform with phases ω^2 and ω^{-2}. All other fields are invariant under the $\mathbb{Z}_N$.

In the dimensional reduction of the the $(2,0)$ theory on T^4, the $SU(2)_L$ fields get mapped to (say) left-movers and the $SU(2)_R$ fields get mapped to (say) right-movers. Thus, we see that the orbifold has a chiral action. In particular, the four bosons that arise from $Y_{\alpha\beta}$ give rise to four **left-moving chiral bosons** and the $\psi_{A\alpha}$ give rise to four **left-moving chiral fermions.**

2.6.2 $\mathbb{Z}_N$ Action From the Poincaré Polynomial

Consider the Poincaré polynomial for T^4 weighted by the phases under the $\mathbb{Z}_N$-action for $N = 1, 2, 3, 4$.

$$(1-\omega x)^2(1-\omega^{-1}x)^2 = x^4 - 2x^3\omega - \frac{2x^3}{\omega} + x^2(\omega^2+\omega^{-2}) + \frac{x^2}{\omega^2} + 4x^2 - 2x\omega - \frac{2x}{\omega} + 1 \quad (2.33)$$

In the above expansion, we identify even powers of x with bosons in the $1+1$-dimensional theory and odd powers with fermions. The coefficient multiplying the term gives the orbifold action. Thus six of the bosons are always periodic and the other two have fractional moding determined by the phase.

It appears that one can use the Poincaré polynomial to obtain the $\mathbb{Z}_5$ action on the dual type IIA string. However, that has to be written as a $SO(4)$ action rather than a $SU(2)_L$ subgroup. When $N = 5$, the $\mathbb{Z}_5$ action is best seen by choosing the T^4 to be given by $\mathbb{R}^4/\Gamma_{A_4}$. The eigenvalues of the generator of the $\mathbb{Z}_5$ are given by ω^r, $(r = 1, 2, 3, 4)$. The corresponding Poincaré polynomial is

$$\begin{aligned}(1-\omega x)&(1-\omega^2 x)(1-\omega^{-1}x)(1-\omega^{-2}x)\\ &= 1 + 2x^2 + x^4 + (\omega^3+\omega+1/\omega+1/\omega^3)x^2\\ &\quad - x/\omega - x/\omega^2 - \omega x - \omega^2 x - \omega^2 x^3 - \omega x^3 - x^3/\omega - x^3/\omega^2 \quad (2.34)\end{aligned}$$

We now present the details of the orbifold action (on the left-movers) for the Green-Schwarz type IIA-superstring that we just derived.

[N=2] $\omega = -1$ implies that $\omega^2 = 1$. Thus, one has eight periodic bosons and eight anti-periodic fermions.

[N=3] $\omega = \exp(2\pi i/3)$ One has six periodic bosons and two bosons which pick up phases ω and ω^2. Four fermions go to ω times themselves and the other four go to ω^{-1} times themselves.

[N=4] $\omega = \exp(\pi i/2)$ One has six periodic bosons and two anti-periodic bosons. Four fermions go to ω times themselves and the other four go to ω^{-1} times themselves.

[N=5] $\omega = \exp(2\pi i/5)$ This is different from the other three examples. One ends up with four periodic bosons and the other four change by a phase ω^r $(r = 1, 2, 3, 4)$. The eight fermions break up into two sets of four fermions. Within each set, one fermion picks up a phase ω^r ($r = 1, 2, 3, 4$).

Thus, the second description gives rise to an **asymmetric orbifold** of the type IIA string on T^6 and thus is analogous to CHL compactifications of the heterotic string. Recall that the heterotic string arises as the type IIA NS5-brane wraps $K3$ in the place of T^4 that we considered.

2.6.3 Type II Dyon Degeneracy From Modular Forms

As mentioned in previous section, computing the dyon spectrum is non-trivial because dyons do not appear in the perturbative spectrum of string theory. In fact, dyon counting necessarily requires computing the degrees of freedom coming from the solitonic sector of the theory. The dyon degeneracy formula can be obtained in two different ways, giving rise to either a additive formula or a multiplicative one.

As shown in [2], for the CHL models, there are two modular forms that one constructs – one is the generating function of the dyon degeneracies (denoted by $\widetilde{\Phi}_k(\mathbf{Z})$) and another (denoted by $\Phi_k(\mathbf{Z})$) is the one related to R^2-corrections in the CHL string. Let us call the corresponding modular forms in the type II models to be $\widetilde{\Psi}_k(\mathbf{Z})$ and $\Psi_k(\mathbf{Z})$. The weight k of the Siegel modular form for the type II models is given by

$$k + 2 = \frac{12}{N+1} , \tag{2.35}$$

when $N + 1|12$ i.e., $N = 2, 3, 5$. For $N = 4$, one has $k = 1$.

We will discuss the modular forms $\widetilde{\Psi}_k(\mathbf{Z})$ and $\Psi_k(\mathbf{Z})$ of the type II orbifolds in chapter 6 where we discuss the construction of all modular forms appearing in the this work. Now we turn to the microscopic counting of dyonic states and sketch the computation in the case of the CHL and type II models as shown by David, Jatkar and Sen[16, 31].

2.7 Counting Dyons in $\mathcal{N} = 4$ Supersymmetric Strings

In the rest of this chapter, we will simultaneously discuss both models: the CHL and type II. Consider description 3 where one has type IIB string theory compactified on $\mathcal{M} \times \widetilde{S}^1 \times S^1$ where $\mathcal{M}$ is either $K3$ or T^4. We then take an orbifold of this theory by a $\mathbb{Z}_N$ symmetry. The action of the symmetry group is generated by a transformation g which involves a

$1/N$ unit shift along the circle S^1 together with an order N transformation $\tilde{g}$ in $\mathcal{M}$. The transformation $\tilde{g}$ is chosen such that it commutes with the $\mathcal{N}=4$ supersymmetry generators of the parent theory and hence preserves the $\mathcal{N}=4$ supersymmetry. Our discussion here closely follows the review of Sen[20].

Following the chain of dualities, we have seen that the transformation $\tilde{g}$ gets mapped, in description 2, to a transformation $\widehat{g}$ that acts only as a shift on the right-moving degrees of freedom on the world-sheet and as a shift plus rotation on the left-moving degress of freedom. Description 2 is obtained by taking an asymmetric orbifold of heterotic or type IIA string theory on $T^4 \times \widehat{S} \times S^1$ by a $1/N$ unit of shift along S^1 together with the transformation $\widehat{g}$. All the supersymmetry comes from the right-moving sector of the world-sheet. The field S_H is the axion-dilaton in the second description and to the complex structure modulus of the torus $\widetilde{S}^1 \times S^1$ in the first description.. The matrix valued scalar field M encodes information about the shape and size of the compactification space $\mathcal{M}' \times \widehat{S} \times S^1$. and the components of the NSNS sector 2-form along it[5]. The gauge fields A_μ are related to the ones coming from the dimensional reduction of the ten-dimensional metric, NSNS anti-symmetric tensor field and gauge fields, without any further electric-magnetic duality transformation. The elementary string states carry electric charge $\mathbf{q}_e$, and various solitons carry magnetic charge $\mathbf{q}_m$.

2.7.1 Tracking dyons through dualities

Recall, the $\frac{1}{4}$-BPS dyons possess charges which are mutually non-local and therefore they do not appear in the perturbative spectrum of the theory. The electric charge vector $\mathbf{q}_e$ and the magnetic charge vector $\mathbf{q}_m$ of a state are defined in the second description. We take the coordinate radii of $S^1/\mathbb{Z}_N$ and $\widetilde{S}^1$ to be 1. The radius of S^1 before orbifolding is taken to be N and the $\mathbb{Z}_N$ orbifolding action involves a $2\pi/N$ translation along S^1. The momentum along S^1 is thus quantized in multiples of $1/N$.

We consider the following dyonic configuration in description 3: Q_5 D5-branes wrapped on $\mathcal{M} \times S^1$, Q_1 D1-branes wrapped on S^1, a single Kaluza-Klein monopole associated with the circle $\tilde{S}^1$ with negative magnetic charge, momentum $-k/N$ along S^1 and momentum J along $\tilde{S}^1$. Also, since a D5-brane wrapped on $\mathcal{M}$ carries, besides the D5-brane charge, $-\beta$ units of induced D1-brane charge, where β is given by the Euler character of $\mathcal{M}$ divided by 24, the net D1-brane charge of the system is $(Q_1 - \beta Q_5)$. (β is zero when $\mathcal{M} = T^4$ and 1

[5]In both the CHL and type II models, $\mathcal{M}'$ is a four-torus. In the type II models, the four-torus, $\mathcal{M}'$, is obtained by T-dualizing all circles on $\mathcal{M} = T^4$ and we will denote it by $\widehat{T}^4$.

when $\mathcal{M} = K3$.)

Following the duality chain and using the sign conventions used in[20], one sees that the above configuration in description 3 leads to a different configuration in description 1. Let us replace a (-1)-charged Kaluza-Klein monopole by a single NS5-brane wrapped on $\mathcal{M}' \times S^1$, Q_5 NS5-branes by Q_5 Kaluza-Klein monopoles along $\widehat{S}^1$; J units of momenta along $\widetilde{S}^1$ is replaced by $-J$ fundamental strings winding $\widehat{S}^1$, where $\widehat{S}^1$ is the circle T-dual to $\widetilde{S}^1$. Further, the D1 charge becomes $(-Q_1 + \beta Q_5)$ fundamental strings wrapping on S^1. The $\mathbb{Z}_N$ orbifold action involves $\mathbb{Z}_N$ orbifold of $\mathcal{M}'$ and simultaneous $1/N$ unit of shift along S^1. Since the orbifolded circle is not participating in the T-duality transformation, the orbifold action commutes with the T-duality transformation.

Finally, one carries out a string-string duality to arrive at the second description[6]. Under this action, all fundamental strings are replaced by NS5-branes and vice versa. Thus, in the end we have Q_1 Kaluza-Klein monopoles along $\widehat{S}^1$, $(-Q_1+\beta Q_5)$ NS5 wrapping $\mathcal{M}' \times S^1$, $-k/N$ units of momentum along S^1, $-J$ NS5 -branes wrapping $\mathcal{M}' \times \widehat{S}^1$, Q_1 NS5-branes wrapping $\mathcal{M}' \times S^1$, and a single fundamental string wrapping S^1. The result is summarized in Figure 2.3.

Description 3	chain of dualities $\longrightarrow$	Description 2
Q_5 D5's wrapping $\mathcal{M} \times S^1$		Q_5 KK monopole for $\widehat{S}^1$
Q_1 D1's wrapping S^1		$(-Q_1+\beta Q_5)$ NS5 wrapping $\mathcal{M}' \times S^1$
momentum $-k/N$ along S^1		momentum $-k/N$ along S^1
momentum J along $\widetilde{S}^1$		$-J$ NS5 wrapping $\mathcal{M}' \times \widehat{S}^1$
-1 KK monopole for $\widetilde{S}^1$		one fund. string wrapping S^1

Figure 2.3: Tracking Dyon configurations. When $\mathcal{M} = K3$, $\mathcal{M}' = T^4$ and $\beta = 1$. When $\mathcal{M} = T^4$, then $\mathcal{M}' = \widehat{T}^4$ and $\beta = 0$.

The second description exclusively contains description in terms of fundamental strings, NS5-branes, Kaluza-Klein monopoles and momenta. If we denote momenta along $S^1 \times \widehat{S}^1$ by $\vec{n}$, fundamental string winding charges along them by $\vec{w}$ and NS5-brane, and Kaluza-Klein monopole charges by $\vec{N}$ and $\vec{W}$ respectively then the T-duality invariants constructed from these electric and magnetic charges are

$$\mathbf{q}_e^2 = 2\vec{n} \cdot \vec{w}\,, \quad \mathbf{q}_m^2 = 2\vec{N} \cdot \vec{W}\,, \quad \mathbf{q}_e \cdot \mathbf{q}_m = \vec{n} \cdot \vec{N} + \vec{w} \cdot \vec{W}\,. \tag{2.36}$$

[6] We follow the conventions followed in [20]

It is easy to check that these T-duality invariants take the following values *before* the orbifold action,

$$\mathbf{q}_e^2 = 2k\,, \quad \mathbf{q}_m^2 = 2Q_5(Q_1 - \beta Q_5)\,, \quad \mathbf{q}_e \cdot \mathbf{q}_m = J\,. \tag{2.37}$$

The $\mathbb{Z}_N$ orbifold action commutes with the entire duality chain and is therefore well defined in any description ('duality frame'). It is convenient for us to discuss it in the second description so that we can easily read out its effect on dyonic charges. The $\mathbb{Z}_N$ orbifold acts by $1/N$ shift along S^1, which results in reducing the circle radius by factor of N. Thus fundamental unit of momentum along S^1 is N and hence momentum along S^1 in the orbifolded theory becomes n/N. To maintain J NS5-branes transverse to S^1 after the orbifold we need to start with N copies of J NS5-branes symmetrically arranged on S^1 before orbifold. The resulting configuration has

$$\frac{1}{2}\mathbf{q}_e^2 = 2k/N\,, \quad \frac{1}{2}\mathbf{q}_m^2 = Q_1 Q_5\,, \quad \mathbf{q}_e \cdot \mathbf{q}_m = J\,, \tag{2.38}$$

in the orbifolded theory.

The S-duality symmetry of this theory in the second description is related to the T-duality symmetry in the original type IIB description. The $1/N$ shift along S^1 breaks the S-duality symmetry of the second description to $\Gamma_1(N)$.

2.8 Microscopic Counting of Dyonic States

In this subsection, we will discuss the microscopic counting of dyon degeneracies carried out by David and Sen[16]. The dyonic configuration corresponds to the BMPV black hole at the center of Taub-NUT space[49].The main idea used by David-Sen is to use the 4D-5D correspondence combined with known dualities to map the counting of states in this configuration to the counting of dyonic degeneracies in the CHL string.

Let $d(\mathbf{q}_e, \mathbf{q}_m)$ denote the number of bosonic minus fermionic $\frac{1}{4}$-BPS supermultiplets carrying a given set of charges $(\mathbf{q}_e, \mathbf{q}_m)$ in the configuration described in the previous section. The dyonic charges of the configuration when $Q_5 = 1$ are given by

$$\mathbf{q}_e^2 = 2k/N\ ,\ \mathbf{q}_m^2 = 2(Q_1 - \beta)\ ,\ \mathbf{q}_e \cdot \mathbf{q}_m = J\ . \tag{2.39}$$

The quantum numbers k and J can arise from three different sources:

1. The excitations of the Kaluza-Klein monopole carrying momentum $-l_0'/N$ along S^1.

2. The overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole carrying momentum $-l_0/N$ along S^1 and j_0 along $\widetilde{S}^1$.

3. The motion of the Q_1 D1-branes in the worldvolume of the D5-brane carrying momentum $-L/N$ along S^1 and J' along $\widetilde{S}^1$.

Thus, we have

$$l_0' + l_0 + L = k \ , \quad j_0 + J' = J \ . \tag{2.40}$$

So, in the weak coupling limit, one can ignore the interaction between the three different sets of degrees of freedom and obtain the generating function of dyonic degeneracies of the whole system as a product of the generating functions of each of the three separate pieces. Let $f(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{\nu})$ denote the generating function of the whole system:

$$f(\rho, \sigma, \nu) = \sum_{k, Q_1, J} d(\mathbf{q}_e, \mathbf{q}_m) \ e^{2\pi i \left[\sigma(Q_1 - 1)/N + \rho k + \nu J\right]} \ . \tag{2.41}$$

Then, from the above argument it is given by

$$f\left(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{\nu}\right) = \frac{1}{64} \ e^{-2\pi i \sigma/N} \Big(\sum_{Q_1, L, J'} (-1)^{J'} d_{D1}(Q_1, L, J^{'}) \ e^{2\pi i(\sigma Q_1/N + \rho L + \nu J')} \Big) \Big(\sum_{l_0, j_0} (-1)^{j_0} d_{CM}(l_0, j_0) \ e^{2\pi i l_0 \rho + 2\pi i j_0 \nu} \Big) \Big(\sum_{l_0'} d_{KK}(l_0') \ e^{2\pi i l_0' \rho} \Big). \tag{2.42}$$

where $d_{D1}(Q_1, L, J^{'})$ is the degeneracy of the Q_1 D1-branes moving in the plane of the D5-brane, $d_{CM}(l_0, j_0)$ is the degeneracy associated with the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole(i.e., its motion in Taub-NUT space), and $d_{KK}(l_0')$ is the degeneracy associated with the excitations of the Kaluza-Klein monopole. The factor of 1/64 removes the degeneracy of a single $\frac{1}{4}$-BPS supermultiplet. Let us write $f(\rho, \sigma, \nu)$ as

$$f(\rho, \sigma, \nu) = \left[\hat{\mathcal{E}}_{S^*(K3/\mathbb{Z}_N)}(\rho, \sigma, \nu) \times \mathcal{E}_{\rm TN}(\rho, \nu) \times g(\rho)\right]^{-1} \ . \tag{2.43}$$

where

$$
\begin{aligned}
\left[\mathcal{E}_{S^*(K3/\mathbb{Z}_N)}(\rho,\sigma,\nu)\right]^{-1} &\equiv \sum_{Q_1,L,J'} (-1)^{J'} d_{D1}(Q_1,L,J')\; e^{2\pi i(\sigma Q_1/N+\rho L+\nu J')}\,, \\
\left[\mathcal{E}_{\mathrm{TN}}(\rho,\nu)\right]^{-1} &\equiv \frac{1}{4}\sum_{l_0,j_0}(-1)^{j_0} d_{CM}(l_0,j_0)\; e^{2\pi i l_0\rho+2\pi i j_0\nu}\,, \\
\left[g(\rho)\right]^{-1} &\equiv \frac{1}{16}\sum_{l_0'} d_{KK}(l_0')\; e^{2\pi i l_0'\rho}\,.
\end{aligned}
$$

Justin, Jatkar, and Sen carried out the explicit counting of the above partition functions and found that the dyonic degeneracies are generated by an automorphic form which for $N=1$ is the unique weight 10 automorphic form of the modular group $Sp(2,\mathbb{Z})$ which is the same one obtained by DVV. For $N>1$ of the orbifolding group $\mathbb{Z}_N$, however, they found the dyonic degeneracies are generated by other modular forms. The general form of their result for the generating function of the degeneracies of $\frac{1}{4}$-BPS states is

$$
\begin{aligned}
\widetilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{\nu}) &= \exp(2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{\gamma}\widetilde{\sigma}+\widetilde{\nu})) \\
&\times \prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k\in\mathbb{Z}+\frac{r}{N},\\ l\in\mathbb{Z},j\in 2\mathbb{Z}+b\\ k,l\geq 0,j<0 \text{ for } k=l=0}} \left[1-\exp\{2\pi i(\widetilde{\sigma}k+\widetilde{\rho}l+\widetilde{\nu}j)\}\right]^{\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_b^{(r,s)}(4kl-j^2)}\,, \qquad (2.44)
\end{aligned}
$$

where the coefficients $c_b^{(r,s)}$ are defined through the twisted elliptic genera as we will see below. We will shortly compute the above equation explicitly by computing each of the pieces in (2.42).

2.8.1 Counting States of the Kaluza Klein Monopole

We first count the degeneracy of the half-BPS states associated with the excitations of the Kaluza-Klein monopole carrying momentum $-l_0/N$ along S^1. Type IIB string theory compactified on $\mathcal{M}\times\widetilde{S}^1\times S^1$ in the presence of a Kaluza-Klein monopole can be described by type IIB string theory in the background $\mathcal{M}\times TN\times S^1$ where TN denotes Taub-NUT space described by the metric

$$
ds^2=\Big(1+\tfrac{R_0}{r}\Big)\Big(dr^2+r^2(d\theta^2+sin^2\theta d\phi^2)\Big)+R_0^2\Big(1+\tfrac{R_0}{r}\Big)^{-1}\Big(2d\xi+cos\theta d\phi\Big)^2, \qquad (2.45)
$$

where R_0 denotes the size of the Taub-NUT space. We take a $\mathbb{Z}_N$ orbifold of this theory generated by the transformation g. The Taub-NUT space breaks eight of the sixteen supersymmetries in type IIB on $K3$ and quantization of its fermionic zero modes gives rise to a multiplicative factor of $16 = 2^{8/2}$. Following the chain of dualities, one sees that the Taub-NUT space gets mapped to the heterotic string wrapped on a $\mathbb{Z}_N$-orbifold of the heterotic string. The degeneracy $d_{KK}(l_0')$ corresponds to the degeneracy of the heterotic string in a twisted sector. Thus, $g(\rho/N)$ is *the partition function of the heterotic string (in a twisted sector) with the supersymmetric right-movers in their ground state.* Hence, it can also be identified with the generating function of degeneracies of electrically charged $\frac{1}{2}$-BPS states. The BPS condition requires that the right-Fmoving oscillators are in their ground state for both $\mathcal{M} = K3$ (the CHL models) and $\mathcal{M} = T^4$ (the type II models).

We are looking for the number of ways a total momentum $-l_0/N$ along S^1 can be partitioned into g-invariant modes. Part of this momentum comes from the momentum of the Kaluza-Klein monopole vacuum without any excitations and this is calculated by mapping the Kaluza-Klein monopole to a fundamental string state in a dual description of the theory.

To count the g-invariant modes, we have to first determine the spectrum of the massless fields in the world volume theory of the Kaluza-Klein monopole solution, and the transformations of the various fields under the action of the orbifold group generator $\widetilde{g}$. From this, we can determine all the g-invariant modes on the Kaluza-Klein monopole, by noting that a field that picks up a $\widetilde{g}$ phase $e^{2\pi\imath k/N}$ must carry momentum $n - k/N(n, k, \in \mathbb{Z})$ along S^1, so that the phase obtained due to the translation along S^1 cancels the $\widetilde{g}$ phase.

We begin by analyzing the spectrum of the theory. First, we concentrate on the bosonic fields. There are 8 non-chiral, right-moving masless scalar fields coming as follows: three come from the oscillations in the three transverse directions of the Kaluza-Klein monopole. Two come from the reduction of the 2-form field of type IIB string theory along the harmonic 2-form of the Taub-NUT space. Reduction of the self-dual four form field of type IIB string theory along the tensor product of the harmonic 2-form of the Taub-NUT space and a harmonic 2-form on $\mathcal{M}$ gives rise to a chiral scalar field on the world-volume. The chirality of the scalar field depends on whether the harmonic 2-form on $\mathcal{M}$ is self-dual or anti-self-dual. Thus, in the case of T^4 we get 3 right moving scalars, and 3 left moving scalars. For $K3$, we get three right-moving scalars, and 19 left-moving scalars.

The fermionic fields come from the Goldstino fermions associated wth broken supersymmetry generators. For the case of $\mathcal{M} = T^4$ theory, there are 32 unbroken supersymmetry charges of which 16 are broken in the presence of the Taub-NUT space. Of the 16 remaining

Goldstino fermions on the world-volume of the Kaluza-Klein monopole, 8 are right-moving and the remaining 8 are left-moving, since type IIB string theory is non-chiral. For the case of $\mathcal{M} = K3$ theory, there are 16 unbroken supersymmetries, of which 8 are broken in the presence of the Taub-NUT space. The remaining 8 Goldstino fermion fields associated with the broken supersymmetry transformation are right-moving, since according to our convention the 8 unbroken supersymmetry transformation parameters on S^1 are left-chiral.

Putting it all together we see that the spectrum of the world-volume theory of the Kaluza-Klein monopole consists of 8 bosonic and 8 fermionic right-moving massless fields. In addition, for $\mathcal{M} = T^4$, it has 8 left-moving bosonic and 8 left-moving fermionic fields, while for $\mathcal{M} = K3$, the world-volume theory has 24 left-moving massless bosonic fields and no left-moving fermionic fields.

Next, we have to work out the $\widetilde{g}$ transformation properties of the various modes. The problem of studying the $\widetilde{g}$ transformation properties of the left-moving bosonic and fermionic degrees of freedom, it can be shown, reduces to the problem of studying the action of the $\widetilde{g}$ action on the even and odd harmonic forms of $K3$. The net action of the $\widetilde{g}$ on the 8) left-handed scalar fields is given by the action of $\widetilde{g}$ on the 8 even degree harmonic forms of $\mathcal{M}$, while its action on the left-moving fermions can be represented by the action of $\widetilde{g}$ on the 1- and 3−forms of $\mathcal{M}$. The difference between the number of even and odd degree harmonic forms, weighted by $\widetilde{g}$, is equal to $Q_{0,s}$. Thus, the number of left-handed bosons minus fermions carrying a $\widetilde{g}$ quantum number $e^{2\pi\imath l s/N}$ is given by

$$n_l = \tfrac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi\imath l s/N} Q_{0,s} = \sum_{s=0}^{N-1} e^{-2\pi\imath l s/N}\left(c_0^{0,s}(0) + 2c_1^{(0,s)}\right) \tag{2.46}$$

where the last equality comes from the expression for $Q_{0,s}$ in terms of the coefficients $c_b^{(r,s)}$.

We must now determine the spectrum of the BPS excitations of the Kaluza-Klein monopole, which is obtained by taking the tensor product of the irreducible 16-dimensional supermultiplet with either fermionic or bosonic excitations involving the left-moving degrees of freedom on the world-volume of the Kaluza-Klein monopole. Let d_{KK} denote the degeneracy of states associated with the left-moving oscillator excitations carrying total momentum $-l_0'/N$, weighted by $(-1)^{F_L}$. To calculate $d_{KK}(l_0')$ we need to count the number of ways the total momentum $-l_0'/N$ can be distributed among the different oscillators, there being n_l

oscillators carying momentum $-l/N$. This gives

$$\sum_{l_0'} d_{KK}(l_0') e^{2\pi\imath\widetilde{\rho}l_0'} = 16 e^{2\pi\imath NC\widetilde{\rho}} \prod_{l=1}^{\infty} (1 - e^{2\pi\imath\widetilde{\rho}l})^{-n_l}, \tag{2.47}$$

where the factor of 16 comes from the fermionic zero modes. The constant C represents the $-l_0'/N$ quantum number of the vacuum of the Kaluza-Klein monopole when all the oscillators are in their ground state and is equal to

$$C = -\widetilde{\alpha}/N, \tag{2.48}$$

where $\widetilde{\alpha}$ is given in terms of $Q_{r,s}$ by

$$\widetilde{\alpha} = \frac{1}{24N} Q_{0,0} - \frac{1}{2N} \sum_{s=1}^{N-1} Q_{0,s} \frac{e^{2\pi\imath s/N}}{(1 - e^{2\pi\imath s/N})^2}. \tag{2.49}$$

Putting it all together, we get

$$g(\widetilde{\rho}) \equiv \sum_{-l_0'} d_{KK}(-l_0') e^{2\pi\imath\widetilde{\rho}l_0'} = 16 e^{-2\pi\imath\widetilde{\alpha}\widetilde{\rho}} \prod_{l=1}^{\infty} (1 - e^{2\pi\imath\widetilde{\rho}l})^{-\sum_{s=0}^{N-1} e^{2\pi\imath ls/N} (c_0^{(0,s)}(0) + c_1^{(0,s)}(-1))} \tag{2.50}$$

2.8.2 Counting States Associated With the Relative Motion of the D1-D5 System

To compute d_{D1}, which counts the states associated with the motion of the D1-brane in the plane of the D5-brane, we start by considering a single D1-brane moving inside a D5-brane. We analyze the world-volume theory of a single $D1$-brane inside a $D5$-brane. In the weak coupling limit the dynamics of the D1-brane inside a D5-brane is insensitive to the presence of the Kaluza-Klein monopole, the two-dimensional theory describing this system has a $(4,4)$ supersymmetry. Consider a D1-brane wrapping along the direction in which $S^1/\mathbb{Z}_N$ has period 2π. Let σ denote the coordinate along the length of the D1-brane and w the winding number of the D1-brane along $S^1/\mathbb{Z}_N$, then σ changes by $2\pi w$ when we traverse the whole length of the string, while the physical coordinate of the D1-brane shifts by $2\pi r$ along S^1 where r and w are related as

$$r = w \mod N . \tag{2.51}$$

In the target space $\mathcal{M}$ under $\sigma \to \sigma + 2\pi w$ the location of the D1-brane gets transformed by $\widetilde{g}^r = \widetilde{g}^w$. Thus the target space $\mathcal{M}$ is subject to the above condition and the states will be twisted by $\widetilde{g}^r$. Further, since the supersymmetry generators are required to commute with $\widetilde{g}$, the supercurrents will satisfy periodic boundary condition under $\sigma \to \sigma + 2\pi w$. Since the D1-brane has coordinate length $2\pi w$, the momentum along S^1 can be identified as the $(\bar{L}_0 - L_0)/w$ eigenvalue of this state. And since, the BPS condition forces $\bar{L}_0$ to vanish, a total momentum $-l/N$ corresponds to a state with

$$L_0 = lw/N, \quad \bar{L}_0 = 0\ . \tag{2.52}$$

In the presence of the Kaluza-Klein monopole background a transition ϵ along S^1 must be accompanied by a rotation 2ϵ in $U(1)_L \subset SU(2)_L$. Let us denote by F_L and F_R, twice the $U(1)_L \subset SU(2)_L$ and $U(1)_R \subset SU(2)_R$ generators respectively. F_L is the world-sheet fermion number associated with the left-moving sector of the $(4,4)$ superconformal field theory, while F_R is the world-sheet fermion number associated with the right-moving sector. The total world-sheet fermion number $F_L + F_R$ can be interpreted as the space-time fermion number from the point of view of a five-dimensional observer at the center of Taub-NUT space.

The quantum number j is the F_L eigenvalue of the state. The four and five-dimensional statistics differ by a factor of $(-1)^j$ and hence, in counting the total number of bosonic minus fermionic states weighted by $(-1)^j$ with a given set of charges, we must compute the number of states weighted by $(-1)^{F_L+F_R}$. And, finally we must pick only states which are $\mathbb{Z}_N$-invariant. Since the total momentum along S^1 is $-l/N$, the state picks up a phase $e^{-2\pi i l/N}$ under a 2π translation. Thus the projection operator onto $\mathbb{Z}_N$ invariant states is given by

$$\frac{1}{N}\sum_{s=0}^{N-1} e^{2\pi \imath s l/N}\widetilde{g}^s. \tag{2.53}$$

Putting it all together, we get for the total number of $\mathbb{Z}_N$ invariant bosonic minus fermionic states weighted by $(-1)^j$ of the single D1-brane carrying quantum numbers w, l, j is given by

$$n(w,l,j) \equiv \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi \imath s l/N}\mathrm{Tr}_{RR,\widetilde{g}^r}(\widetilde{g}^s(-1)^{F_L+F_R}\delta_{NL_0,lw}\delta_{F_L,j}), \quad r = w \bmod N, \tag{2.54}$$

where the $Tr_{RR,\widetilde{g}^r}$ denotes trace over the RR sector states twisted by $\widetilde{g}^r$ in the superconformal σ-model with target space $K3$.

In terms of the coefficients $c_b^{(r,s)}$, $n(w,l,j)$ is given by

$$n(w,l,j) = \sum_{s=0}^{N-1} e^{-2\pi\imath sl/N} c_b^{(r,s)}(4lw/N - j^2), \quad r = w \bmod N, \quad b \equiv j \bmod 2. \tag{2.55}$$

Using this result for a single D1-brane spectrum we need to find the spectrum of multiple D1-branes moving inside the D5-brane. Let the total D1-brane charge be W, and total momentum along S^1 and $\widetilde{S}^1$ be $-L/N$ and J' respectively. Let us denote by $d_{D1}(W,L,J')$ the total number of bosonic minus fermionic states of the whole system, weighted by $(-1)^{J'}$ which represents the number of ways of distributing the quantum numbers W, L and J' into individual D1-branes carrying quantum numbers (W_i, l_i, j_i) subject to the constraint

$$W = \sum_i w_i, \quad L = \sum_i l_i, \quad J' = \sum_i j_i, \quad w_i, l_i, j_i \in \mathbb{Z}, \quad w_i \geq 1, \quad l_i \geq 0. \tag{2.56}$$

A straightforward combinatoric analysis gives

$$\sum_{W,L,J'} d_{D1}(W,L,J')(-1)^{J'} e^{2\pi\imath(\widetilde{\sigma}W/N + \widetilde{\rho}L + \widetilde{\nu}J')} = \prod_{w,l,j\in\mathbb{Z}; w>0, l\geq 0} (1 - e^{2\pi\imath(\widetilde{\sigma}w/N + \widetilde{\rho}l + \widetilde{\nu}j)})^{-n(w,l,j)}. \tag{2.57}$$

In terms of the coefficients $c_b^{(r,s)}$ this takes the form

$$\sum_{W,L,J'} d_{D1}(W,L,J')(-1)^{J'} e^{2\pi\imath(\widetilde{\sigma}W/N + \widetilde{\rho}L + \widetilde{\nu}J')}$$
$$= \prod_{r=0}^{N-1} \prod_{b=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, \\ j\in 2\mathbb{Z}+b; k'>0, l\geq 0}} \left(1 - e^{2\pi\imath(\widetilde{\sigma}k' + \widetilde{\rho}l + \widetilde{\nu}j)}\right)^{-\sum_{s=0}^{N-1} e^{2\pi\imath sl/N} c_b^{(r,s)}(4lk' - j^2)} \tag{2.58}$$

This is the partition function for states associated with the motion of the D1-branes in the plane of the D5-brane.

2.8.3 Counting States Associated With the Overall Motion of the D1-D5 System

The overall motion of the D1-D5 system has two components – the center of mass motion of the D1-D5 system along the Taub-NUT space transverse to the plane of the $D5$-brane, and the dynamics of the Wilson lines on the $D5$-brane along $\mathcal{M}$. Of these, the first component is independent of the choice of $\mathcal{M}$, while the second exists only if $\mathcal{M}$ has non-contractible one cycles, i.e. for $\mathcal{M} = T^4$. We analyze each in turn now starting with the center of mass motion of the D1-D5 motion in Taub-NUT space.

Dynamics of the D1-D5 Motion in Taub-NUT Space

The contribution from this is independent of the the choice of $\mathcal{M}$. When the transverse space is Taub-NUT, the low energy dynamics of the system can be described by a $(1+1)$-dimensional supersymmetric field theory. The world-volume theory is a sum of two mutually non-interacting pieces – a theory of free left-moving fermions and an interacting theory of scalars and right-moving fermions. From the point of view of a five-dimensional observer sitting at the center of Taub-NUT space, the D1-D5 system in the Taub-NUT target space is described by a set of four free left-moving $U(1)_L$ invariant fermion fields, together with an interacting theory of four bosons and four right-moving $U(1)_L$ non-invariant fermions. The two bosons and two of the right-moving fermions carry a j_0 quantum number 1, while the other two bosons and right-moving fermions carry a j_0 quantum number of -1. The unbroken supersymmetry transformations act only on the scalars and the right-moving fermions. All the fields carry integral momenta along S^1.

To compute the partition function, first consider the free left-moving fermions which carry only l_0 quantum numbers but no j_0 quantum numbers. Their contribution is given by

$$Z_{\text{free}}(\tilde{\rho}) \equiv \text{Tr}_{\text{free left-moving fermions}}((-1)^F(-1)^{j_0}e^{2\pi i\tilde{\rho}l_0}e^{2\pi i\tilde{\nu}j_0}) = 4\prod_{n=1}^{\infty}(1-e^{2\pi i n N\tilde{\rho}})^4, \qquad (2.59)$$

where F is the total contribution to the space-time fermion number, except from the fermion zero-modes associated with the broken supersymmetry generators, from the point of view of an asymptotic four-dimensional observer. The factor of 4 comes from the quantization of the free fermion zero modes.

Next we compute the partition function for the part that is interacting. There are two parts to this, the zero mode oscillators and the non-zero modes. By taking the size R_0 of the Taub-NUT space to be large so that the metric is almost flat, and in a local region of the Taub-NUT space the world-volume theory of the D1-D5 system is almost free. Then we can compute the contribution due to the non-zero mode bosonic and fermionic oscillators by placing the D1-D5 system at the origin of the Taub-NUT space and treating them as oscillators of free fields. Further, we need to examine only the left-moving bosonic oscillators carrying momentum $-l_0/N$ along S^1 and angular momentum j_0, since the right-moving bosonic and fermionic oscillators are in their ground state. The contribution to the partition function from these oscillators is

$$\begin{aligned} Z_{\text{osc}}(\tilde{\rho},\tilde{\nu}) &\equiv \text{Tr}_{\text{oscillators}}((-1)^F(-1)^{j_0}e^{2\pi i\tilde{\rho}l_0}e^{2\pi i\tilde{\nu}j_0}) \\ &= \prod_{n=1}^{\infty}\frac{1}{(1-e^{2\pi inN\tilde{\rho}+2\pi i\tilde{\nu}})^2(1-e^{2\pi inN\tilde{\rho}-2\pi i\tilde{\nu}})^2}, \end{aligned} \tag{2.60}$$

where we use the fact that since these oscillators are bosonic from the five-dimensional point of view, they have statistics $(-1)^F=(-1)^j_0$ from the four-dimensional point of view.

Finally we have to evaluate the partition function for the zero-mode oscillators of the interacting part of the theory. Since there are four bosonic and four fermionic fields, we can think of it as the dynamics of a superparticle, with four bosonic and four fermionic coordinates, which transform in a pair of spinor representations, moving in the Taub-NUT space. The partition function for these modes is give by

$$Z_{\text{zero}}(\tilde{\nu}) \equiv \text{Tr}_{\text{zero modes}}((-1)^F(-1)^{j_0}e^{2\pi i\tilde{\nu}j_0} = -\sum_{j_0=1}^{\infty} j_0 e^{2\pi i\tilde{\nu}j_0} = -\frac{e^{2\pi i\tilde{\nu}}}{(1-e^{2\pi i\tilde{\nu}})^2}\ . \tag{2.61}$$

Putting together all the constituent partition functions, the partition function associates with the center of mass motion of the D1-D5 system in the Taub-NUT space is given by

$$\begin{aligned} \sum_{l_0,j_0} d_{\text{transverse}} \quad & (l_0,j_0)(-1)^j_0 e^{2\pi i l_0\tilde{\rho}+2\pi i j_0\tilde{\nu}} = Z_{\text{free}}(\tilde{\rho})Z_{\text{osc}}(\tilde{\rho}\tilde{\nu})Z_{\text{zero}}(\tilde{\nu}) \\ = \quad & -4e^{2\pi i n\tilde{u}}(1-e^{2\pi i\tilde{\nu}})^{-2} \\ & \times\prod_{n=1}^{\infty}\{(1-e^{2\pi inN\tilde{\rho}})^4(1-e^{2\pi inN\tilde{\rho}+2\pi i\tilde{\nu}})^{-2}(1-e^{2\pi inN\tilde{\rho}-2\pi i\tilde{\nu}})^{-2}\}, \end{aligned} \tag{2.62}$$

2.8.4 The Dynamics of Wilson Lines Along $\mathcal{M}$

We now turn to the contribution to the partition function from the dynamics of the Wilson lines along $\mathcal{M} = T^4$. We can ignore the presence of the Kaluza-Klein monopole and the D1-branes and consider the dynamics of the D5-brane wrapped on $T^4 \times S^1$. However, the Kaluza-Klein monopole will be used in the identification between the angular momentum carried by the system from the point of view of the five-dimensional observer at the center of the Taub-NUT space and the momentum along the circle S^1 from the point of view of the asymptotic four-dimensional observer. Taking the T^4 to have small size we can regard the world-volume theory of the D5-brane as $(1+1)$-dimensional which contains eight scalars associated with four Wilson lines and four transverse coordinates and 16 massless fermions of which eight are left-moving and eight are right-moving. We have to considering only the supersymmetry generators that commute with $\widetilde{g}$. The $\widetilde{g}$ transformation mixes the scalars associated with the coordinates transverse to the D5-brane with the eight of the sixteen fermions on the D5-brane world-volume and mixes the scalars associated with the Wilson lines with the other eight fermions. We hve already counted the contribution to the partition function from the transverse coordinates and their superpartners in (2.62), hence here need only consider the world-volume fields consisting of the Wilson lines and their superpartners. Since the $\widetilde{g}$ invariant supersymmetry generators are non-chiral, so are the superpartners of the Wilson line. There are four left-moving and four right-moving fermionic such fields. Of these, only the left-moving oscillators contribute, since the right-moving oscillators are in their ground state when we work in the background of the Kaluza-Klein monopole. Thus, there are only four bosonic and four fermionic left-moving modes. Invariance under $\widetilde{g}$ requires that two of the four bosonic modes carrying momentum along S^1 be of the form $k + \frac{1}{N}$ while the other two be of the form $k - \frac{1}{N}$. Neither has any momentum along $\widetilde{S}^1$. Similarly, two of the fermionic modes carry momentum $k + \frac{1}{N}$ along S^1 while the other two carry $k - \frac{1}{N}$. These modes, however, carry ± 1 units of momentum along $\widetilde{S}^1$. As before, the statistics of the oscillators are altered by a factor of $(-1)^j_0$ as we come down from four to five-dimensions. Thus, if $d_{\text{Wilson}}(l_0, j_0)$ denotes the number of bosonic minus fermionic states associated with these modes carrying a total momentum l_0/N along S^1 and j_0 along $\widetilde{S}^1$, then

$$\sum_{l_0,j_0} d_{\rm Wilson}(l_0,j_0)(-1)^{j_0} e^{2\pi i l_0\tilde\rho+2\pi i j_0\tilde\nu}$$
$$= \prod_{l\in N\mathbb{Z}+1,l>0}(1-e^{2\pi i l\tilde\rho})^{-2}\prod_{l\in N\mathbb{Z}-1,l>0}(1-e^{2\pi i l\tilde\rho})^{-2}\prod_{l\in N\mathbb{Z}+1,l>0}(1-e^{2\pi i l\tilde\rho+2\pi i\tilde\nu})$$
$$\prod_{l\in N\mathbb{Z}+1,l>0}(1-e^{2\pi i l\tilde\rho-2\pi i\tilde\nu})\prod_{l\in N\mathbb{Z}-1,l>0}(1-e^{2\pi i l\tilde\rho+2\pi i\tilde\nu})\prod_{l\in N\mathbb{Z}-1,l>0}(1-e^{2\pi i l\tilde\rho-2\pi i\tilde\nu}) \quad (2.63)$$

The full partition function of the overall dynamics of the D1-D5 system is given by the product of the partition functions (2.62) and (2.63) of the dynamics of the transverse modes and (for $\mathcal{M}=T^4$) of the Wilson lines along T^4. The final result can be written compactly using the coefficients $c_b^{(r,s)}(u)$, and noting that

$$c_1^{(0,s)}(-1) = \begin{cases} \frac{2}{N} & \text{for } \mathcal{M}=K3 \\ \frac{1}{N}(2-e^{2\pi i s/N}-e^{-2\pi i s/N}) & \text{for } \mathcal{M}=T^4\ . \end{cases} \quad (2.64)$$

The product of (2.62) and (for $\mathcal{M}=T^4$) (2.63) can be written as

$$\sum_{l_0,j_0} d_{\rm CM}(l_0,j_0)(-1)^{j_0} e^{2\pi i l_0\tilde\rho+2\pi i j_0\tilde\nu} = -4e^{-2\pi i\tilde\nu}\prod_{l=1}^{\infty}(1-e^{2\pi i l\tilde\rho})^{2\sum_{s=0}^{N-1}e^{-2\pi i l s/N}c_1^{(0,s)}}$$
$$\prod_{l=1}^{\infty}(1-e^{2\pi i l\tilde\rho+2\pi i\tilde\nu})^{-\sum_{s=0}^{N-1}e^{-2\pi i l s/N}c_1^{(0,s)}}\prod_{l=1}^{\infty}(1-e^{2\pi i l\tilde\rho-2\pi i\tilde\nu})^{-\sum_{s=0}^{N-1}e^{-2\pi i l s/N}c_1^{(0,s)}} \quad (2.65)$$

for both $\mathcal{M}=K3$ and $\mathcal{M}=T^4$.

2.8.5 The Full Partition Function

Using (2.42), (2.50), (2.58) and (2.65) we put together the full partition function:

$$f(\widetilde\rho,\widetilde\sigma,\widetilde\nu) = e^{-2\pi i(\widetilde\alpha\widetilde\rho+\widetilde\nu)}\prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k\in\mathbb{Z}+\frac{r}{N},l\in\mathbb{Z},\\ j\in 2\mathbb{Z}+b\\ k,l\geq 0,\\ j<0 \text{ for } k=l=0}}(1-e^{2\pi i(\widetilde\sigma k+\widetilde\rho l+\widetilde\nu j)})^{-\sum_{s=0}^{N-1}e^{-2\pi i l s/N}c_b^{(r,s)}(4kl-j^2)} \quad (2.66)$$

The multiplicative factor $e^{-2\pi i(\widetilde\alpha\widetilde\rho+\widetilde\nu)}$ and the $k=0$ term in the expression come from the terms involving $d_{CM}(l_0,j_0)$ and $d_{KK}(l_0')$. Comparing with the expression for $\widetilde\Phi(\widetilde\rho,\widetilde\sigma,\widetilde\nu)$ in

(2.44) we can rewrite (2.66) as

$$f\left(\widetilde{\rho},\widetilde{\sigma},\widetilde{\nu}\right)=\frac{e^{2\pi i\widetilde{\gamma}\widetilde{\sigma}}}{\widetilde{\widetilde{\Phi}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{\nu}}) \ . \tag{2.67}$$

The degeneracy $d(\mathbf{q}_e,\mathbf{q}_m)$ is given by

$$d(\mathbf{q}_e,\mathbf{q}_m)=(-1)^{\mathbf{q}_e\cdot\mathbf{q}_m+1}\frac{1}{N}\int_{\mathcal{C}}d\widetilde{\rho}d\widetilde{\sigma}d\widetilde{\nu}e^{-\pi i(N\widetilde{\rho}\mathbf{q}_e^2+\widetilde{\sigma}\mathbf{q}_m^2/N+2\widetilde{\nu}\mathbf{q}_e\cdot\mathbf{q}_m)}\frac{1}{\widetilde{\widetilde{\Phi}}\left(\widetilde{\rho},\widetilde{\sigma},\widetilde{\nu}\right)} \ . \tag{2.68}$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by $(\widetilde{\rho},\widetilde{\sigma},\widetilde{\nu})$, given by

$$\widetilde{\rho}_2=M_1,\quad \widetilde{\sigma}_2=M_2\quad \widetilde{\nu}_2=-M_3,$$
$$0\leq\widetilde{\rho}_1\leq 1,\quad 0\leq\widetilde{\sigma}_1\leq N,\quad 0\leq\widetilde{\nu}_1\leq 1 \ .$$

Here M_1, M_2 and M_3 are large but fixed positive numbers with $M_3 << M_1, M_2$. The M_i's are determined from the requirement that the Fourier expansion is convergent in the region of integration. Thus, we have computed the degeneracy formula by explicitly counting the black hole microstate. This completes the discussion on the counting of the black hole microstates.

2.9 Walls of Marginal Stability

We conclude this chapter with a discussion on the walls of marginal stability. For a given set of charges, the moduli space will be divided into connected domains where the $\frac{1}{4}$-BPS states are stable and the degeneracy formula is valid[33]. As one moves around in the moduli space, there arises the possibility of some of the $\frac{1}{4}$-BPS states to decay into smaller constitutents. In that case the degeneracy formula will not remain valid when we go into a region where some of the $\frac{1}{4}$-BPS states that were present earlier have decayed. The degeneracy formula will, obviously, change to reflect this change in the number of $\frac{1}{4}$-BPS states with the given charges. The regions in moduli space where the degeneracy formula is valid are bounded by codimension one subspaces on which the BPS state under consideration becomes marginally stable and the spectrum changes discontinuously across these subspaces. These codimension one subspaces in moduli space are called the ***walls of marginal stability***. The jump in degeneracy occurs through a subtle dependence of the contour on moduli[33].
The walls of marginal stability in the axion-dilaton plane (modelled by the upper-half plane

with coordinate λ) is the real codimension one subspace across which one $\frac{1}{4}$-BPS state decays into a pair of $\frac{1}{2}$-BPS states[33](see also[50, 51]). Consider the following decay of a torsion one $\frac{1}{4}$-BPS dyon into two $\frac{1}{2}$-BPS dyons

$$\begin{pmatrix} \mathbf{q}_e \\ \mathbf{q}_m \end{pmatrix} \longrightarrow \begin{pmatrix} ad\ \mathbf{q}_e - bd\ \mathbf{q}_m \\ ca\ \mathbf{q}_e - cb\ \mathbf{q}_m \end{pmatrix} \oplus \begin{pmatrix} -bc\ \mathbf{q}_e + bd\ \mathbf{q}_m \\ -ac\ \mathbf{q}_e + ad\ \mathbf{q}_m \end{pmatrix} , \tag{2.69}$$

where the kinematics of the decay imply that the integers a, b, c, d are such that[33]

1. $ad - bc = 1$.
2. The equivalence relation $(a, b, c, d) \sim (a\sigma^{-1}, b\sigma^{-1}, c\sigma, d\sigma)$ with $\sigma \neq 0$.
3. Exchanging the two decay products implies the equivalence under:
$$(a, b, c, d) \rightarrow (c, d, -a, -b).$$
4. Charge quantization requires $ad, bd, bc \in \mathbb{Z}$ and $ac \in N\mathbb{Z}$.

One can show that by suitable use of the equivalences given above, one can always choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ for $N = 2, 3, 4$. In the upper-half plane, these walls are circular arcs determined by the equation[33, 50]

$$\left[\mathrm{Re}(\lambda) - \tfrac{ad+bc}{2ac}\right]^2 + \left[\mathrm{Im}(\lambda) + \tfrac{\mathcal{E}}{2ac}\right]^2 = \tfrac{1+\mathcal{E}^2}{4a^2c^2} , \tag{2.70}$$

where $\mathcal{E}$ is a real function of all other moduli M. It is easy to see that the arcs intersect the real λ axis at the points $\frac{b}{a}$ and $\frac{d}{c}$ for any $\mathcal{E}$. When $\mathcal{E} = 0$, the arcs are semi-circles centred on the real λ-axis with radius $\frac{1}{2ac}$. When $\mathcal{E} \neq 0$, the center of the circle moves into the interior of the upper half plane with radius also increasing – all this with the intercepts on the real axis remaining unchanged. When either $a = 0$ or $c = 0$, the circles become straight lines perpendicular to the real axis for $\mathcal{E} = 0$ and making a suitable angle for $\mathcal{E} \neq 0$. The sole effect on non-zero $\mathcal{E}$ is to 'deform' the semi-circles into circular arcs, so we restrict the discussion to the case when $\mathcal{E} = 0$.

A fundamental domain is constructed by first restricting the value of $\mathrm{Re}(\lambda)$ to the interval $[0, 1]$. The straight lines $\mathrm{Re}(\lambda) = 0, 1$ correspond to two walls of marginal stability. Next, one looks for the largest semi-circle with one end at $\lambda = 0$ on the real axis that is compatible with the quantization of charges. This semi-circle intersects the real axis at some point in the interval $[0, 1]$ – this turns out to be at $1/N$. The procedure is then (recursively) repeated by looking for another semi-circle with one end at $1/N$ till one hits the mid-way point $1/2$. A similar procedure is done starting with the largest semi-circle with one end on the point $\lambda = 1$ on the real axis. One obtains the following set of points for $N = 1, 2, 3$:

$$\left(\tfrac{0}{1}, \tfrac{1}{1}\right) , \quad \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right) , \quad \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right) . \tag{2.71}$$

A fundamental domain is then given by restricting to the region bounded by these semi-circles and the two walls connecting $\lambda = 0, 1$ to infinity. The two straight lines may be included by adding the 'points' $\frac{-1}{0}$ and $\frac{1}{0}$. The fundamental domains are given in Figure 2.4.

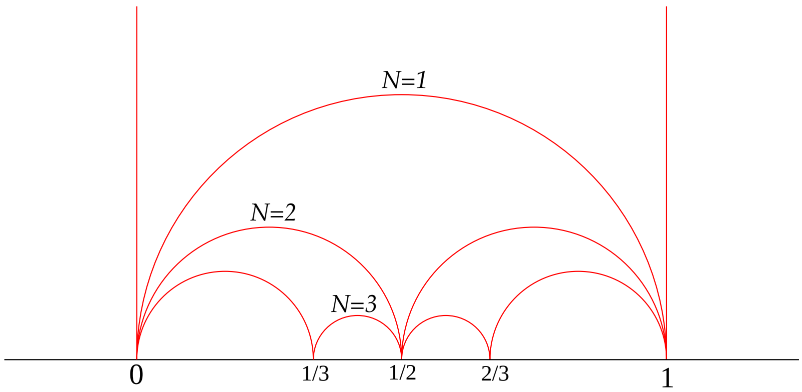

Figure 2.4: Fundamental domains for the $N = 1, 2, 3$ CHL models. We will later see that the same region appears as the Weyl chamber of a BKM Lie superalgebra in each case.

For $N > 3$, this picture does not terminate – one needs an infinite number of semi-circles to obtain a closed domain. For $N = 4$, the following sequence is obtained on (using Sen's method)

$$\left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \ldots, \tfrac{-2n+1}{-4n}, \tfrac{-n}{-2n-1}, \ldots, \tfrac{1}{2}, \ldots, \tfrac{n+1}{2n+1}, \tfrac{2n+1}{4n} \ldots, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right) . \tag{2.72}$$

Let α_n denote the semi-circle with intercepts $\left(\frac{2n-1}{4n}, \frac{n}{2n+1}\right)$ and β_n the semi-circle with intercepts $\left(\frac{n+1}{2n+1}, \frac{2n+1}{4n}\right)$ for all $n \in \mathbb{Z}$. Note that α_0 and β_0 represent the two straight lines at $\mathrm{Re}(\lambda) = 0, 1$ respectively. The fundamental domain corresponding to the above sequence is depicted in Figure 2.5. It may be thought of as a regular polygon with infinite edges with the infinite-dimensional dihedral group as its symmetry group, $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$. $D_\infty^{(1)}$ is generated by two generators: a reflection y and a shift γ given by:

$$y: \quad \alpha_n \to \alpha_{-n}\ ,\ \beta_n \to \beta_{-n-1} \quad \text{and} \quad \gamma : \alpha_n \to \alpha_{n+1}\ ,\ \beta_n \to \beta_{n-1}\ , \tag{2.73}$$

satisfying the relations $y^2 = 1$ and $y\cdot\gamma\cdot y = \gamma^{-1}$. There is a second $\mathbb{Z}_2$ generated by δ defined as follows:

$$\delta: \quad \alpha_n \longleftrightarrow \beta_n\ . \tag{2.74}$$

The transformations (γ, δ) generate another dihedral symmetry $D_\infty^{(2)}$.

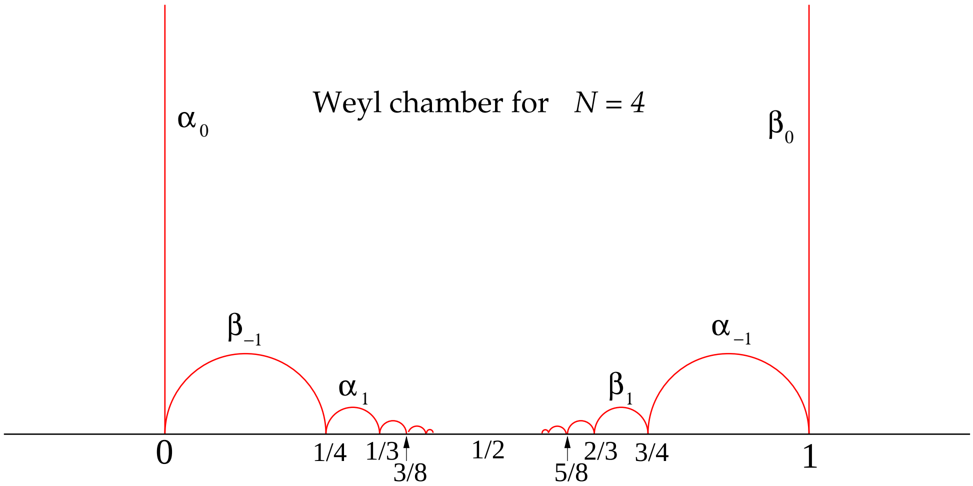

Figure 2.5: The fundamental domain for $N = 4$ CHL model is bounded by an infinite number of semi-circles as the BKM Lie superalgebra has infinite real simple roots. Each of the semi-circles indicated represent real simple roots that appear with multiplicity one in the sum side of the denominator formula. Note that the diameter of the semi-circle is reducing as one gets closer to $\frac{1}{2}$. The point $\frac{1}{2}$ is approached as a limit point of the infinite sequence of semi-circles. We will later see that the same region appears as the Weyl chamber of a BKM Lie superalgebra that we construct.

This completes our discussion of the walls of marginal stability of the CHL orbifolds. Later, when we study BKM Lie superalgebras related to the CHL models, we will see that the walls of marginal stability are related to the walls of the Weyl chamber of the corresponding BKM Lie superalgebras.

2.10 Conclusion and Remarks

In this chapter we have looked at the problem of counting dyons in $\mathcal{N} = 4$ supersymmetric string theories. The degeneracy of the dyonic states are generated by modular forms. We will explore the structure of these modular forms in later chapters. We will construct these modular forms by different methods in chapter 5 and study their algebraic side in chapter 6. We also studied the walls of marginal stability for the $\frac{1}{4}$-BPS states. We will later see how these are related to the algebraic structure coming from the modular forms.

3

BKM Lie Algebras

3.1 Introduction

The starting point, both historically and pedagogically, for this chapter is the theory of finite-dimensional semi-simple Lie algebras classified by Cartan and Killing. The classification lets one study Lie algebras generically, rather than on a case by case basis. It also gives an overview, and hence suggestive directions for generalizations in both the internal structure of Lie algebras and the theory of finite-dimensional semi-simple Lie algebras in general. However, in our study of finite-dimensional semi-simple Lie algebras, their classification is not our primary interest – generalization to infinite-dimensional Lie algebras is.

Our main application, in this thesis is the subject of Borcherds-Kac-Moody (BKM) Lie superalgebras which occur in the degeneracy formula of $\frac{1}{4}$-BPS dyons in the CHL models. These are obtained via generalizations of finite-dimensional semi-simple Lie algebras to their infinite-dimensional counterparts. Our aim in this chapter will be to give a quick and modest introduction to BKM Lie superalgebras, for which we start with a brief exposition of finite-dimensional semi-simple Lie algebras with a veiw towards understanding the generalizations that give BKM Lie superalgebras. It is with this skewed perspective that we will choose and discuss the topics in this chapter. After introducing the theory of finite-dimensional semi-simple Lie algebras, we treat BKM Lie superalgebras as the generalization of the finite-dimensional algebras and include affine and Kac-Moody Lie algebras as special cases of them. We will use examples to bridge the gap in theory and intuition incurred by this leap in pedagogy.

The chapter is organized as follows. We start with the basic definitions of complex semi-simple Lie algebras and study their representation theory to introduce the notions of the

Cartan subalgebra, roots, weights, the Weyl group etc. and understand what role these play in the structure of the Lie algebras and how one can classify all the finite-dimensional semi-simple Lie algebras from the knowledge of these notions. The main idea is to get an intuitive feeling of these ideas in the context of examples which are easier to understand. It will not be possible to introduce all the constructions needed to rigorously define the same notions in the infinite-dimensional case, and hence it is simpler to understand them by extending the intiution built in the context of the simpler finite-dimensional cases. Although we discuss the representation theory of general semi-simple Lie algebras, we relegate the discussion on the denominator identities to the end of the chapter since this is the most important idea for us. The discussion on the denominator identities of all classes of Lie algebras, both finite and infinite-dimensional, is given in one place so that it is easier to understand each in relation to the other and also note the important differences amongst them. Next we discuss the infnite-dimensional Lie algebras building on the ideas introduced in the finite-dimensional setting.

For the case of BKM Lie superalgebras, given how much generality the class of Lie algebras encompass, it would take a lot more technical setting to rigorously introduce the notions mentioned above. We do not make such an attempt here. Most of the definitions are given as an extension to the intuition developed in the finite-dimensional and affine settings. Introducing any more structure would be more confusing than illuminating. The example of the fake monster Lie algebra is discussed to help understand the concepts (like roots, imaginary simple roots, multiplicities, the denominator identity, etc.) developed in the context of BKM Lie superalgebras.

It must be mentioned at the very outset, that it is beyond the scope of this work to give even a semi-complete discussion of BKM Lie superalgebras for the subject is both vast, and intricate. As mentioned above, there are certain ideas (the denominator identity) that we need a lot in the problem we address in the next chapters, and it is these ideas that we will try to motivate and understand. Rather than motivate these ideas precisely and pedantically, we will try to understand their origins intuitively starting from their analogs in the finite-dimensional semi-simple Lie algebras and ending with an example of a BKM Lie superalgebra. This chapter is based mostly on [52, 53, 54, 55, 3, 4, 56]. The reader is also encouraged to see [57, 58, 59, 60, 61, 62, 63, 64] for BKM Lie superalgebras in relation to string theory.

3.2 Definition and Properties

A Lie algebra can be understood in relation to a Lie group, whose algebra it is (as the algebra of left-invariant smooth vector fields on the group), or just as an algebra over a field satisfying certain additional axioms. Both the notions ultimately describe the same object, they only appear motivated from different points of view. From the narrow point of view of this chapter, to digress into the theory of Lie groups and understand Lie algebras from them would serve us no purpose, so we just define a Lie algebra as an algebra over a field.

Definition 3.2.1 *An* ***algebra*** *is a vector space over a field* $\mathbb{K}$ *(which is* $\mathbb{C}$ *for all our purposes) endowed with a product* $[.,.] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *that is distributive over addition and compatible with scalar multiplication by elements of* $\mathbb{K}$*. It is a* ***Lie algebra*** *if, in addition, it also has the following properties*

(i) $[.,.]$ *is bilinear,*

(i) ***Antisymmetry*** *:*

$$[x, x] = 0, \quad \forall x \in \mathfrak{g} \quad (\text{ and hence } [x, y] = -[y, x], \quad \forall x, y \in \mathfrak{g}), \tag{3.1}$$

(ii) ***Jacobi identity****:*

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \quad \forall x, y, z \in \mathfrak{g} \ . \tag{3.2}$$

Examples of Lie algebras in physics should be familiar from the study of the theory of angular momentum. All the finite-dimensional Lie algebras we will study will be matrix Lie algebras, that is, they can be understood as a subalgebra of $gl(n, \mathbb{C})$, which is the associative algebra of all $(n \times n)$ matrices over $\mathbb{C}$.

The more interesting Lie algebras, and the ones we will be dealing with, are a sub-class of the above definition called semi-simple Lie algebras whose definition we next motivate

Definition 3.2.2 *A* ***Lie subalgebra*** $\mathfrak{a}$ *of* $\mathfrak{g}$ *is a subspace satisfying* $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$.

It is also a Lie algebra. A Lie subalgebra $\mathfrak{a}$ is called ***abelian*** if $[\mathfrak{a}, \mathfrak{a}] = 0$. One such subalgebra, called the Cartan subalgebra, will play an important role in understanding the structure of semi-simple Lie algabras.

Definition 3.2.3 *A subspace* $\mathfrak{a}$ *of* $\mathfrak{g}$ *is called an* ***ideal*** *if it satisfies* $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$.

An ideal, by its definition, is also a Lie subalgebra. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a Lie algebra, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]$.

Definition 3.2.4 *A finite-dimensional Lie algebra* $\mathfrak{g}$ *is said to be* ***simple*** *if it is non-abelian (i.e.* $[\mathfrak{g}, \mathfrak{g}] \neq 0$*) and* $\mathfrak{g}$ *has no proper non-zero ideals (i.e. its only ideals are* $\mathfrak{g}$ *and* 0*). A finite-dimensional Lie algebra* $\mathfrak{g}$ *is said to be* ***semi-simple*** *if* $\mathfrak{g}$ *is isomorphic to a direct sum of simple Lie algebras.*

One can also define a semi-simple Lie algebra, equivalently, through its Killing form. The Killing form is a symmetric bilinear form on $\mathfrak{g}$. There is also a third, and equivalent, definition of semi-simple Lie algebras in terms of the Chevalley-Serre relations. It is via this definition that it is simplest to pass to the infinite-dimensional case from finite-dimensional semi-simple Lie algabras. We will introduce the Chevalley generators, and the Chevalley-Serre relations they satisfy, at the appropriate juncture. Here we give the definition of semi-simplicity through the Killing form.

For any Lie algebra we can define a linear map ad: $\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}\mathfrak{g}$, called the adjoint mapping, given by

$$\mathrm{ad}_x(y) = [x, y] \ , \tag{3.3}$$

where $\mathrm{End}_{\mathbb{C}}\mathfrak{g}$ is the space of all $\mathbb{C}$-linear maps from $\mathfrak{g}$ to $\mathfrak{g}$.

We can now define the ***Killing form*** on $\mathfrak{g}$. Given two elements x and y in $\mathfrak{g}$, we can define a linear transformation $(\mathrm{ad}_x \mathrm{ad}_y)$ from $\mathfrak{g}$ to itself. The Killing form of $\mathfrak{g}$ is given by

$$B(x, y) = \ \mathrm{Tr}(\mathrm{ad}_x \ \mathrm{ad}_y) \ . \tag{3.4}$$

The Killing form is invariant in the sense that

$$B([x, y], z) = B(x, [y, z]) \ .$$

We can now define an equivalent definition of a finite-dimensional semi-simple Lie algebra called **Cartan's criterion for semi-simplicity**.

Theorem 3.2.5 *A Lie algebra* $\mathfrak{g}$ *is semi-simple if and only if the Killing form on* $\mathfrak{g}$ *is non-degenerate.*

The reason for the multiple definitions is that when we pass from finite-dimensional semi-simple Lie algebras to more general Lie algebras, some of the definitions are more suitable generalizations than others, and hence it is helpful to understand the definition from different points of view. We give some examples of Lie algebras, before moving on to the idea of representations.

3.2.1 Examples

The most familiar example of a group from physics is the three-dimensional rotation group $SO(3)$. It is the group of all rotations about the origin on the three-dimensional Euclidean space, $\mathbb{R}^3$, with composition as the group operation. It represents the symmetries of a sphere. As a matrix group, it is the group of 3×3 real matrices A such that $A^{\mathrm{T}}A = I$, and $\det A = 1$. The Lie algebra associated to this is the space of 3×3 complex matrices satisfying $X^{\mathrm{T}} = -X$, denoted by $so(3)$, where the superscript T denotes the transpose of a matrix.

It is a specific example of a more general class of Lie algebra $so(n,\mathbb{C})$, known as the ***special orthogonal*** Lie algebra, which is the space of all $n\times n$ complex matrices satisfying $X^{\mathrm{T}} = -X$.

Another familiar example is the ***special linear*** Lie algebra, denoted $sl(n,\mathbb{C})$. It is the space of all $n\times n$ complex traceless matrices over $\mathbb{C}$. The algebra $sl(2,\mathbb{C})$ is the algebra of 2×2 traceless matrices over $\mathbb{C}$. We had earlier mentioned the Virasoro algebra satisfied by the Fourier modes L_m of the energy momentum tensor $T_{\alpha\beta}$. We will see in the course of this chapter that the Virasoro algebra is related to the algebra $sl(2,\mathbb{C})$ as its "loop algebra".

Another example is the symplectic algebra $sp(n)$. It is the space of $2n\times 2n$ complex matrices X such that $JX^{\mathrm{T}}J = X$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We will see that the symplectic group plays a very important role in the theory of Siegel modular forms that occur in the dyon degeneracy formulae.

All the above matrix algebras are subalgebras of the Lie algebra $gl(n,\mathbb{C})$, known as the general linear algebra, which is the space of all $n\times n$ complex matrices. They are all simple Lie algebras as well.

3.3 Representations

We will now study representations of Lie algebras. We start by explaining the idea of representing a group or algebra on a vector space. The simplest example is of the just cited rotation group $SO(3)$ which acts on the three-dimensional Euclidean space $\mathbb{R}^3$. If one wanted to realize an action of the group on a different space, say a vector space V of dimension d, one cannot obviously use the 3×3 matrices to act on this space. We need linear operators of the right dimension to define a sensible action on the space V. We also need their action on V to be such that the group action of $SO(3)$ is faithfully captured on V. Thus, what we need is a map from $SO(3)$ to the space of invertible linear operators that are of the appropriate dimension to act on the space V and are such that they *represent* the action of $SO(3)$ on V. This is the idea of a representation of a group. Let us now make this idea more precise starting with a formal definition of the representation of a Lie algebra.

Definition 3.3.1 *Let V be a vector space over a field $\mathbb{C}$, and let $\mathfrak{g}$ be a Lie algebra. A* ***finite-dimensional representation*** *of $\mathfrak{g}$ on V is a continuous homomorphism ρ of Lie algebras $\rho : \mathfrak{g} \to End_{\mathbb{K}}V$. ρ has to be $\mathbb{C}$-linear and has to satisfy*

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x) \quad \textit{for all } x, y \in \mathfrak{g} \ . \tag{3.5}$$

We will (as do most authors) call V the representation when we mean the representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}V$. A subspace W of V is called ***invariant*** if $\rho(g)w \in W$ for all $w \in W$ and all $g \in \mathfrak{g}$. A representation with no non-trivial invariant subspaces is called *irreducible*. The dimension of the representation is defined to be the dimension of the vector space V.

The best way to understand the theory of representations of a group or algebra is by looking at some examples. We will study the representations of the Lie algebras $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$ in detail shortly, but before that we can cite a few examples which exist for any Lie algebra.

We have already seen implicitly an example of a representation in eq.(3.3) – that of the adjoint mapping. Recall that it was defined as the map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}\mathfrak{g} \ . \tag{3.6}$$

given by the formula

$$\mathrm{ad}_x(y) = [x,y] \ . \tag{3.7}$$

Comparing (3.6) with (3.3), we see that 'ad' is a representation where the space V is taken to be $\mathfrak{g}$. 'ad' is a Lie algebra homomorphism and is, therefore a representation of $\mathfrak{g}$, called the ***adjoint representation***. It is the representation of the Lie algebra $\mathfrak{g}$ acting on itself.

Another representation that exists for all Lie algebras is the trivial representation. If $\mathfrak{g}$ is a Lie algebra of $n \times n$ matrices over $\mathbb{C}$ then the trivial representation $\rho : \mathfrak{g} \to gl(1,\mathbb{C})$ is given by

$$\rho(x) = 0 \ ,$$

for all $x \in \mathfrak{g}$. This is (obviously) an irreducible representation.

3.3.1 The Irreducible Representations

The irreducible representations of $sl(2, \mathbb{C})$, apart from their well known relevance to physics, are very illuminating in understanding the origins of various ideas that we will study to understand general semi-simple Lie algebras. It will also help in understanding the idea, of representing a Lie algebra on an arbitrary vector space, introduced above, in a concrete way. It is also the simplest non-trivial example of a semi-simple Lie algebra, yet a very important one. From a physics point of view, $sl(2, \mathbb{C})$ is the complexification of $su(2)$ and every finite-dimensional complex representation of $su(2)$ extends to a complex linear representation of $sl(2, \mathbb{C})$. Also since $su(2) \cong so(3)$, the study of whose representation are of physical significance, the study of the representations of $sl(2, \mathbb{C})$ also have a physical motivation.

Our purpose in studying the representations of $sl(2, \mathbb{C})$ and $sl(3, \mathbb{C})$ will be, besides giving examples of representations, to use them to illustrate important aspects of representation theory of semisimple Lie algebras in general. To that end, we will study the irreducible representations of $sl(2, \mathbb{C})$ illustrating the explicit construction of linear operators representing the algebra on an arbitrary vector space, the action of the operators on the representation space, the idea of raising and lowering operators, and the idea of the highest weight, which will later develop into the highest weight theorem for general semi-simple Lie algebras. Also, general complex semi-simple Lie algebras are built out of many copies of $sl(2, \mathbb{C})$, and studying $sl(2, \mathbb{C})$ is preliminary to understanding the representations of semi-simple Lie algebras in general.

In studying the Lie algebra $sl(3, \mathbb{C})$ we will concentrate more on learning about root systems of semi-simple Lie algabras. From this we will also learn about Cartan matrices, Dynkin diagrams and the classification of finite-dimensional semi-simple Lie algebras. We will introduce the representation theory of $sl(3, \mathbb{C})$ with a view towards using it to generalize the notions from $sl(2, \mathbb{C})$ to general semi-simple Lie algabras, via $sl(3, \mathbb{C})$. We will also use it to understand the Weyl group, the character and denominator formulae of Lie algebras.

3.3.2 The Irreducible Representations of $sl(2,\mathbb{C})$

We fix the following basis for $sl(2,\mathbb{C})$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;,\; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;. \tag{3.8}$$

which have the commutation relations[1]

$$[h,e] = 2e \;, [h,f] = -2f \;, [e,f] = h \;. \tag{3.9}$$

Now, if V is a finite-dimensional complex vector space and A, B and C are operators on V satisfying

$$[A,B] = 2B \;, [A,C] = -2C \;, [B,C] = A \;, \tag{3.10}$$

then, because of the skew symmetry and bilinearity of brackets, the linear map $\rho : sl(2,\mathbb{C}) \to gl(V)$ satisfying

$$\rho(h) = A \;,\; \rho(e) = B \;,\; \rho(f) = C$$

will be a representation of $sl(2,\mathbb{C})$ on V. The operators A, B and C which are of suitable dimension to act on the vector space V, represent $sl(2,\mathbb{C})$.

To construct the irreducible representations of $sl(2,\mathbb{C})$ consider the $(m+1)$-dimensional vector space V_m of homogeneous polynomials in two complex variables with total degree $m (m \geq 0)$. V_m is the space of functions of the form

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \ldots + a_m z_2^m \;, \tag{3.11}$$

with $z_1, z_2 \in \mathbb{C}$ and the a_i's arbitrary complex constants.

For any $x \in \mathfrak{g}$ consider the action on V_m given as follows

$$\rho_m(x) f = -(x_{11} z_1 + x_{12} z_2) \frac{\partial f}{\partial z_1} - (x_{21} z_1 - x_{11}) \frac{\partial f}{\partial z_2} \;, \tag{3.12}$$

which maps V_m to V_m. It is also easy to see that $\rho_m(x)\rho_m(y) f = \rho_m(xy) f$, where the product (xy) is the usual multiplication of matrices. This is a representation of $sl(2,\mathbb{C})$ on the vector

[1] The Lie bracket becomes the commutator in any representation

space V_m. In terms of the basis (3.8) the above formula becomes

$$(\rho_m(h)f)(z) = -z_1\frac{\partial f}{\partial z_1} + z_2\frac{\partial f}{\partial z_2} \ .$$

Thus,

$$\rho_m(h) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} \ . \qquad (3.13)$$

The action of $\rho_m(h)$ on a basis element $z_1^k z_2^{m-k}$ is $\rho_m(h)\ z_1^k z_2^{m-k} = (m-2k)z_1^k z_2^{m-k}$. Thus, we see that $z_1^k z_2^{m-k}$ is an eigenvector for $\rho_m(h)$ with eigenvalue $(m-2k)$. In particular, $\rho_m(h)$ is diagonalizable.

Corresponding to the elements x and y we have the following operators

$$\rho_m(x) = -z_2\frac{\partial}{\partial z_1}\ , \quad \rho_m(y) = -z_1\frac{\partial}{\partial z_2}\ ,$$

so that

$$\begin{aligned} \rho_m(x)z_1^k z_2^{m-k} &= -k\ z_1^{k-1} z_2^{m-k+1}\ , \\ \rho_m(y)z_1^k z_2^{m-k} &= (k-m)\ z_1^{k+1} z_2^{m-k-1}\ . \end{aligned} \qquad (3.14)$$

Notice that, since all the basis vectors $z_1^k z_2^{m-k}$ are eigenvectors of $\rho(h)$, knowing the action of $\rho(h)$ on the basis vectors gives V_m as the direct sum of its weight spaces. The representation ρ_m is an irreducible representation of $sl(2,\mathbb{C})$ and there is one such for each integer $m \geq 0$. The representation ρ_m has dimension $m+1$. Any two irreducible representations of $sl(2,\mathbb{C})$ with the same dimension are equivalent.

Given the commutation relations between the elements h, x and y as given above, it is easy to see that the corresponding operators act on an eigenvector u of $\rho(h)$ as follows

$$\begin{aligned} \rho(h)\rho(x)u &= (\alpha+2)\rho(x)u\ , \\ \rho(h)\rho(y)u &= (\alpha-2)\rho(y)u\ . \end{aligned} \qquad (3.15)$$

Since we are working over an algebraically closed field, $\mathbb{C}$, the above equation says that given an eigenvector u of $\rho(h)$, either $\rho(x)u = 0$ (resp. $\rho(y)u = 0$) or $\rho(x)u$ (resp. $\rho(y)u$) is an eigenvector for $\rho(h)$ with eigenvalue $\alpha+2$ (resp. $\alpha-2$). More generally

$$\begin{aligned} \rho(h)\rho(x)^n u &= (\alpha+2n)\rho(x)^n u \ , \\ \rho(h)\rho(y)^n u &= (\alpha-2n)\rho(y)^n u \ . \end{aligned} \tag{3.16}$$

The operators $\rho(x)$ and $\rho(y)$ are called the ***raising*** and ***lowering*** operators respectively, since they have the effect of, respectively, raising and lowering the eigenvalue of $\rho(h)$ by 2. The operator $\rho(x)$ operating on a finite-dimensional vector space can have only finitely many distinct eigenvalues, the operation of raising the eigenvalue by applying the $\rho(x)$ operator cannot be repeated indefinitely and must terminate after a finite number of operations. Thus there will exist some integer $N \geq 0$ such that

$$\rho(x)^N u \neq 0 \ ,$$

but

$$\rho(x)^{N+1} u = 0 \ .$$

Defining $u_0 = \rho(x)^N u$ and $v = \alpha + 2N$, the above equations can be written as

$$\begin{aligned} \rho(h)u_0 &= v\ u_0 \ , \\ \rho(x)u_0 &= 0 \ . \end{aligned} \tag{3.17}$$

v is the highest eigenvalue of $\rho(h)$ in the given representation, and any further operation by $\rho(x)$ gives 0. To the vector u_0 one can apply the operator $\rho(y)$ to lower its eigenvalue by 2. Defining $u_k = \rho(y)^k u_0$, for $k \geq 0$ we have

$$\begin{aligned} \rho(h)u_k &= (v-2k)u_k \ , \\ \rho(x)u_k &= [kv - k(k-1)]u_{k-1} \quad . \end{aligned} \tag{3.18}$$

Again, the operation of lowering cannot be repeated indefinitely, and the u_k's cannot all be non-zero. There must, therefore, exist a non-negative integer m such that

$$u_k = \rho(y)^k u_0 \neq 0 \ ,$$

for all $k \leq m$, but

$$u_{m+1} = \rho(y)^{m+1} u_0 = 0 \ .$$

That is, the eigenvalues of $\rho(h)$ are bounded both from above and below. Now, if $u_{m+1} = 0$, then $\rho(x)u_{m+1} = 0$. Therefore, we have

$$0 = \rho(x)u_{m+1} = [(m+1)v - m(m+1)]u_m = (m+1)(v-m)u_m \ .$$

Since $u_m \neq 0$ and $m+1 \neq 0$, we must have $v = m$, where m is a non-negative integer.

Thus, any finite-dimensional irreducible representation of $sl(2,\mathbb{C})$ acting on a space V, will be of the following form

$$\begin{aligned} \rho(h)u_k &= (m-2k)u_k \ , \\ \rho(y)u_k &= u_{k+1} \ , \\ \rho(y)u_m &= 0 \ , \\ \rho(x)u_k &= [km - k(k-1)]u_{k-1} \ , \\ \rho(x)u_0 &= 0 \ . \end{aligned} \tag{3.19}$$

and vice versa. The vectors $u_0, \ldots, u_m$ will be independent, since they are eigenvectors of $\rho(h)$ with distinct eigenvalues.

Any complex-linear representation of $sl(2,\mathbb{C})$ on a finite-dimensional complex vector space V is completely ***reducible*** in the sense that there exist invariant subspaces $U_1, \ldots, U_r$ of V such that $V = U_1 \oplus \cdots \oplus U_r$ and such that the restriction of the representation to each U_i is irreducible.

In general the irreducible representations of a Lie algebra $\mathfrak{g}$ need not be so conspicuously simple. Two representations may be isomorphic, but the isomorphism may not be immediately apparent. We need to have an invariant property associated to a representation that can save us the need of explicitly veryfying the equivalence/inequivalence of two given representations by writing down the explicit description of the representations in terms of matrices. This leads us to the concept of the ***character*** of a representation which we will study after we look at the representations of $sl(3,\mathbb{C})$.

Before we move on and study the Lie algebra $sl(3,\mathbb{C})$ and its representations, and generalize further to any general semi-simple Lie algebra, we will note down some of the important take-away points which will be important in tracing the origins of the generalizations. We will label them by roman letters and will refer to them wherever this property is involved later in the chapter in connection with $sl(3,\mathbb{C})$ or general semi-simple Lie algebras.

(**A**) The element h plays a special role, in that representations are labelled by the eigenvalues of $\rho(h)$, called *weights*. Every irreducible representation is the direct sum of its weight spaces.

(**B**) Every eigenvalue of $\rho(h)$ is an integer.

(**C**) The eigenvalues are bounded from above and below and the smallest eigenvalue is the negative of the largest. For each weight m, there is a corresponding vector with weight $-m$.

(**D**) The multiplicity of an eigenvalue k equals the multiplicity of $-k$.

(**E**) The operators $\rho(x)$ and $\rho(y)$, respectively, raise and lower the eigenvalues of $\rho(h)$ by 2.

(**F**) If there exists a non-zero element w of V such that $\rho(x)w = 0$ and $\rho(h)w = \mu w$, then there is a non-negative integer m such that $\mu = m$ and the vectors $w, \rho(y)w, \ldots, \rho(y)^m w$ are linearly independent and their span is an irreducible invariant subspace of dimension $m + 1$.

(**G**) If ρ is an $(m + 1)$-dimensional irreducible representation of $sl(2, \mathbb{C})$, then the highest eigenvalue m of ρ is an integer.

(**H**) Going the other way, for every non-negative integer m there exists an irreducible representation of $sl(2, \mathbb{C})$.

(**I**) Any two irreducible representations of $sl(2, \mathbb{C})$ of dimension $(m + 1)$ are equivalent.

We will see how each of the above ideas contains the germs whose generalizations will give us important insights into the theory of semi-simple Lie algebras in general.

3.3.3 The Irreducible Representations of $sl(3, \mathbb{C})$

We will study the general representation theory of semi-simple Lie algebras taking the example of $sl(3, \mathbb{C})$ and taking each idea to its natural generalization to obtain the analogous notions for the case of general semi-simple Lie algebras. Since $sl(3, \mathbb{C})$ is a simple example, it will be easy to see the structure while at the same time not getting bogged down in abstract general theory. Before going to the representation theory of $sl(3, \mathbb{C})$, however, we will study $sl(3, \mathbb{C})$ (and via generalization any semi-simple Lie algebra) in some detail, getting some idea of the structure of the Lie algebras.

We start by choosing a basis for $sl(3,\mathbb{C})$ as follows

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.20}$$

Working out the commutation relations between the various elements, one sees that the span of $\{h_1, e_1, f_1\}$ is a subalgebra of $sl(3,\mathbb{C})$ which is isomorphic to $sl(2,\mathbb{C})$ as is the span of $\{h_2, e_2, f_2\}$. We had earlier mentioned, in motivating the study of representations of $sl(2,\mathbb{C})$, that complex semi-simple Lie algebras are made out of many copies of $sl(2,\mathbb{C})$. All semi-simple Lie algebras are made up of copies of $sl(2,\mathbb{C})$ like the copies of $sl(2,\mathbb{C})$ in $sl(3,\mathbb{C})$ above. This idea holds, with suitable modifications, even for infinite-dimensional Lie algebras. Also note that the elements h_1 and h_2 commute with each other, that is $[h_1, h_2] = 0$.

We will get back to the commutation relations between the other elements in a while, after we introduce the concept of roots and weights. The idea is to get some control over the structure of the Lie algebra.

The broad idea of the programme is as follows. The Cartan subalgebra, as defined above, is abelian and the adjoint action of the Cartan subalgebra on the given semisimple Lie algebra leads to a root-space decomposition of the Lie algebra. The Lie algebras can be studied and classified through their root systems. Using an ordered basis of simple roots, one can construct the Cartan matrix or the equivalent Dynkin diagram which encode the structure of the semi-simple Lie algebra in them. The Weyl group captures the fact that the Cartan matrix and the Dynkin diagram are independent of the choice and ordering of simple roots. Every Cartan matrix arises from a reduced abstract root system, and there is a one-to-one correspondence (upto isomorphism) between the two. This leads to a one-to-one correspondence (upto isomorphism) between complex semi-simple Lie algebras and reduced abstract root systems.

We will use the semisimple Lie algebra $sl(3,\mathbb{C})$ to study and illustrate, and subsequently generalize to general semi-simple Lie algebras, the above notions. We will also study the representation theory of $sl(3,\mathbb{C})$.

3.3.4 Cartan subalgebra, Roots and Weights

Definition 3.3.2 *Given a representation (ρ, V) of $sl(3,\mathbb{C})$, an ordered pair $\mu = (m_1, m_2) \in \mathbb{C}^2$ is called a **weight** for ρ if there exists a vector $v \neq 0$ in V such that*

$$\begin{aligned} \rho(h_1)v &= m_1 v \ , \\ \rho(h_2)v &= m_2 v \ . \end{aligned} \tag{3.21}$$

*A non-zero vector v satisfying the above equation is called a **weight vector** corresponding to the weight μ.*

The space of all vectors satisfying the above conditions (including the zero vector) is called the ***weight space*** corresponding to the weight μ. The dimension of the weight space is called the ***multiplicity*** of the weight. This is generalization of the point **(A)**, from the take-away notes at the end of the last section, where the weights were defined as the eigenvalues of $\rho(h)$. Generalizing the notion to a general semi-simple Lie algebra, one defines a weight as a collection of simultaneous eigenvalues of the $\rho(h_i)$'s which are the set of maximally commuting elements in the Lie algebra. Every representation has atleast one weight, and equivalent representations have the same weights and multiplicities. We will come back to the definition of weights for a general semi-simple Lie algebra later in this section. For now, we continue with $sl(3,\mathbb{C})$.

For $sl(3,\mathbb{C})$, all the weights are of the form $\mu = (m_1, m_2)$ with m_1, m_2 being integers. The weight vectors of the adjoint representation are called ***root vectors***. That is, for a vector z satisfying

$$[h_1, z] = a_1 z \ , \quad [h_2, z] = a_2 z \ ,$$

the pair $\alpha \equiv (a_1, a_2) \in \mathbb{C}^2$ is called a ***root*** and the element z is called the ***root vector*** corresponding to the root α.

The roots (and weights) are defined as the simultaneous eigenvalues of ad_{h_i} $(\rho(h_i))$, where the h_i are the set of maximally commuting elements in the Lie algebra. This set plays a central role in the study of the structure of semi-simple Lie algebras. It is called the ***Cartan subalgebra*** of the Lie algebra. It is defined as follows

Definition 3.3.3 *The **Cartan subalgebra** of a complex semi-simple Lie algebra $\mathfrak{g}$ is the complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties*

(i) For all h_1 and h_2 in $\mathfrak{h}$, $[h_1, h_2] = 0$,

(ii) For all $x \in \mathfrak{g}$, if $[h, x] = 0$ for all $h \in \mathfrak{h}$, then $x \in \mathfrak{h}$,

(iii) For all $h \in \mathfrak{h}$, ad_h is diagonalizable .

Condition (i) says that $\mathfrak{h}$ is a commutative subalgebra of $\mathfrak{g}$, and condition (ii) says that it is *maximally* commutative. It is thus, the normalizer $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} | [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Condition (iii) says each ad_h is diagonalizable, and since all the $h \in \mathfrak{h}$ commute, the ad_h's also commute, and thus they are also diagonalizable *simultaneously*. For the case of $sl(2, \mathbb{C})$ there was only one element, h.

The ***rank*** of a complex semi-simple Lie algebra $\mathfrak{g}$ is defined to be the dimension of its Cartan subalgebra. With the above general definition of the Cartan subalgebra, the roots and root spaces can be defined as follows

Definition 3.3.4 *A **root** of a semi-simple Lie algebra $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h}$) is a non-zero linear functional $\alpha \in \mathfrak{h}^*$ such that there exists a non-zero element $x \in \mathfrak{g}$ with*

$$[h, x] = \alpha(h)x \ ,$$

for all $h \in \mathfrak{h}$.

So, a root is just a (non-zero) collection of simultaneous eigenvalues for the ad_h's. The set of all roots is denoted L. The ***root space*** $\mathfrak{g}_\alpha$ is the space of all $x \in \mathfrak{g}$ for which $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}$. An element of $\mathfrak{g}_\alpha$ is called a ***root vector*** (for the root α).

Going back to the basis elements of $sl(3, \mathbb{C})$ and working out the various commutation relations between the elements, we can now express the same information, using the concept of root vectors, as follows. The vectors x_i and y_i are eigenvectors for h_1 and h_2, and the collection of the eigenvalues are the roots for $sl(3, \mathbb{C})$. Giving the various roots is enough to specify the various commutation relations, which we do below. Here, α denotes the root and $\mathbf{Z}$ the corresponding root vector.

$$\begin{array}{rl} \alpha & \mathbf{Z} \\ (2,-1) & x_1 \\ (-1,2) & x_2 \\ (1,1) & x_3 \\ (-2,1) & y_1 \\ (1,-2) & y_2 \\ (-1,-1) & y_3 \ . \end{array} \tag{3.22}$$

These six roots form a root system conventionally called A_2. We will later come to the various root systems, when we have studied their theory a little more. To carry information about all the roots is redundant, and it is sufficient to only work with two roots from the above six, as the others can be writen in terms of these two. We single out the roots

$$\begin{aligned} \alpha_1 &= (2,-1) \\ \alpha_2 &= (-1,2) \ , \end{aligned} \tag{3.23}$$

and express the other four roots in terms of them. The choice of the labels 1 and 2 is arbitrary, and is equivalent to labelling them the other way. The two roots are called the ***positive simple roots*** and usually denoted Π. The positive simple roots have the property that all the roots can be expressed as linear combinations of the positive simple roots with *integer* coefficients, such that the coefficients are all positive or all negative. More generally, a semisimple Lie algebra of rank r will have r positive simple roots. The positive simple roots are such that for any $\alpha \in L$, we have

$$\alpha = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r \ , \tag{3.24}$$

where the n_j's are integers and either all greater than or equal to zero or all less than or equal to zero (but not all zero simultaneously). Once we fix a set of simple roots, the α's for which $n_j \geq 0$ are called the ***positive roots*** (w.r.t the chosen Π), denoted L_+, and the α's with $n_j \leq 0$ are called the ***negative roots***, denoted L_-. Note that all the elements of Π are in L_+, and are thus called the set of ***positive*** simple roots.

For the case of $sl(3,\mathbb{C})$ all the roots can be expressed in terms of α_1 and α_2 as follows

$$\begin{aligned}
(2,-1) &= \alpha_1 \\
(-1,2) &= \alpha_2 \\
(1,1) &= \alpha_1+\alpha_2 \\
(-2,1) &= -\alpha_1 \\
(1,-2) &= -\alpha_2 \\
(-1,-1) &= -\alpha_1-\alpha_2 \ .
\end{aligned} \tag{3.25}$$

The choice of picking out α_1 and α_2 is, of course, arbitrary. Any other pair which satisfies the above criteria is equally suitable.

Considering the elements of $\mathfrak{h}$ as those belonging to the root space $\mathfrak{g}_0$, we see that the adjoint action of the $h_i \in \mathfrak{h}$ gives a decomposition of the Lie algebra $\mathfrak{g}$ as a direct sum of root spaces(compare with **(A)**), since all the h_i are simultaneously diagonalizable. The Lie algebra $\mathfrak{g}$ can be decomposed as a direct sum as follows

$$\mathfrak{g} = \bigoplus_{\alpha\in L} \mathfrak{g}_\alpha = \bigoplus_{\alpha\in L_-} \mathfrak{g}_\alpha \bigoplus \mathfrak{h} \bigoplus_{\alpha\in L_+} \mathfrak{g}_\alpha \ . \tag{3.26}$$

This means that every element of $\mathfrak{g}$ can be written uniquely as a sum of an element of $\mathfrak{h}$ and one element from each root space $\mathfrak{g}_\alpha$. This is the first sign of emergence of some method to what we have been doing. Now, we begin to see how the decomposition of the Lie algebra as eigenvectors of the ad_h's gives us control over the structure of the Lie algabra. As vector spaces $\mathfrak{g} = N_+ \oplus h \oplus N_-$, where N_+/N_- are the vector spaces generated by the elements with positive and negative eigenvalues w.r.t the Cartan subalgebra h respectively.

The set of simple roots Π of the Lie algebra has all the information of the Lie algebra in it, and as we will see, we can reduce the essence of the problem even further when we relate them to abstract root systems where the whole Lie algebra is captured by a matrix, or an equivalent diagram. These abstract root systems, not just describe for us the Lie algebra whose root systems they are isomorphic to, but in fact will allow us to classify all the possible semisimple Lie algebras into a finite number of classes.

We will look root systems briefly to complete the study we have started with the example of $sl(3,\mathbb{C})$, after we study the Weyl group and Weyl reflections of the root system. Firstly, however, we record some of the properties of the roots, without proof, below.

(i) For any α and β in $\mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$,

(ii) If $\alpha \in \mathfrak{h}^*$ is a root, then so is $-\alpha$ (compare with **(C)**),

(iii) The roots span $\mathfrak{h}^*$,

(iv) If α is a root of $\mathfrak{g}$, then the only multiples of α that are roots are α and $-\alpha$ (compare with **(C)** and **(D)**),

(v) If α and β are roots, the quantity $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ is an integer, where the inner product $\langle .,. \rangle$ is defined as in (3.27).

(vi) For all roots α, the root spaces $\mathfrak{g}_\alpha$ are one-dimensional ,

(vii) For each root α, we can find non-zero elements $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-a}$ and $h_\alpha \in \mathfrak{h}$ such that $x_\alpha, y_{-\alpha}$ and h_α span a subalgebra of $\mathfrak{g}$ isomorphic to $sl(2,\mathbb{C})$.

Having studied the roots and their properties, one can go ahead and consider root systems independent of their origins in semi-simple Lie algebras. This makes sense because, many of the results about root systems involve only the root systems and not the Lie algebras from which they come. Therefore, one can study the theory of root systems on their own. We have already quoted the important results about roots, and digressing into abstract root systems will not serve us any new purpose to devote space to studying them. However, we mention abstract root systems because there is a point to take away from the above study. Given a complex semi-simple Lie algebra, one can associate to it an abstract reduced root system and vice-versa. One can use the classification of the abstract root systems and translate it to classifying semi-simple Lie algebras. The basis of this association is the choice of the Cartan subalgebra, whose simultaneous eigenvalues the roots are. Later when we construct the Cartan matrices for these reduced root systems we will need a particular ordering of the roots (viz. the labelling of α_1 and α_2 mentioned above) and subject to that, there is an isomorphism between abstract root systems and Cartan matrices. We have already seen this happen in the example of $sl(3,\mathbb{C})$, where we singled out two roots as the simple roots, and written the other roots in terms of them and said the ordering of the positive simple roots does not matter. We had also mentioned that the choice of the two roots is arbitrary, and any other set of simple roots that satisfy the proper criteria are equally good candidates. Also, any two Cartan subalgebras of $\mathfrak{g}$ are conjugate to each other. Thus, to make the association between complex semi-simple Lie algebras and Cartan matrices, via root systems, useful we need to examine the independence of the above isomorphisms of the choice of the ordering of the roots.

3.3.5 The Weyl Group

We will study the idea of the Weyl group of $\mathfrak{g}$ now, and in doing so, try to use it to address the above question of the independence of the choice of ordering of simple roots in associating Cartan matrices to complex semi-simple Lie algebras. Before we motivate the idea of the Weyl group, we need a hermitian inner product on $\mathfrak{g}$ which is defined as follows. For matrices x and y, we define an inner product on $\mathfrak{g}$ as

$$\langle x, y\rangle = \mathrm{Tr}(xy^*) \tag{3.27}$$

Consider the set of roots L of a semi-simple Lie algebra $\mathfrak{g}$ and a set, Π, of simple roots that generate L. In the theory of abstract root systems, the set of simple roots is called a ***base***. Let the vector space generated by Π be E. E is just the r-dimensional Euclidean vector space of linear combination of all roots $\alpha \in L$. Note the difference between E and L. Each element $\alpha \in L$ is such that it can be expressed as a linear combination of the elements of Π with integer coefficients and in such a way that the coefficients are either all non-negative or all non-positive, whereas E is just the vector space generated by Π without any such restrictions.

For any two roots $\alpha, \beta \in L$, consider the following linear transformation of E

$$w_\alpha \cdot \beta = \beta - 2\frac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle}\alpha, \quad \beta \in E\ , \tag{3.28}$$

known as a reflection, because geometrically, it is one in the space E as we will explain below. For now, we note the property that $w_\alpha^2 = 1$, $\forall\ \alpha \in L$. For all $\alpha, \beta \in E$, the reflected element $w_\alpha \cdot \beta$ is also a root. The set of all such reflections will thus act as a permutation on the set of roots and taking L to itself. Also, each reflection possesses an inverse (itself), and the composition or reflections is again a reflection (called a 'word'). Thus, the subgroup of the orthogonal group on E generated by all the reflections w_α for $\alpha \in L$ forms a group called the ***Weyl group*** of L and each reflection is known as a ***Weyl reflection***. Since the rank r of $\mathfrak{g}$ is finite, the Weyl group generated as reflections w.r.t the set of positive roots is a finite group. Note that upto this point there has been no mention of the relation to the underlying Lie algebra. In the case when the root system originates from a Lie algebra $\mathfrak{g}$

with Cartan subalgebra $\mathfrak{h}$, the Weyl group is denoted by $\mathcal{W}(\mathfrak{g}, \mathfrak{h})$. We will simply call it $\mathcal{W}$ and the arguments are understood.

Next, we define a quantity known as the ***Weyl vector*** ρ of L as

$$\rho = \tfrac{1}{2} \sum_{\alpha \in L^+} \alpha \ . \tag{3.29}$$

The Weyl vector will play a very important role later when we study the character and denominator formulae of Lie algebras, both finite and infinite-dimensional. Now, looking at the above equation, we notice that the way the Weyl vector is defined, as a sum over the set of roots, may not be suitable for generalization to the case of infinite-dimensional Lie algebras since the root system of infinite-dimensional Lie algebras is not finite. There is an alternate definition which lends itself to generalization to the infinite case without involving infinite sums and we define it below.

Definition 3.3.5 *The* ***Weyl vector****, ρ, of a root system L is defined to be the vector which satisfies*

$$\langle \rho, \alpha_i \rangle = 1, \qquad \textit{for all } \alpha_i \in \Pi \tag{3.30}$$

Eq. (3.30) defines the Weyl vector even for infinite-dimensional Lie algebras and it is this definition that we will use from here on.

We can also develop a geometrical picture of the above ideas in the space E. Let V be a hyperplane through the origin in E such that V does not contain any root. Consider an element α which is perpendicular to this hyperplane. Thus, V will be the set of elements μ in E such that $\langle \alpha, \mu \rangle = 0$, and either side of V will be elements that satisfy the inequality $\langle \alpha, \mu \rangle > 0$ or $\langle \alpha, \mu \rangle < 0$.

Given the root system (L, E), the hyperplane V partitions the space E into two sides. For any vector α in the one-dimensional orthogonal complement of V, let us denote the sides satisfying $\langle \alpha, \mu \rangle > 0$ and $\langle \alpha, \mu \rangle < 0$ by L_+ and L_- respectively. Any element α of L_+ is called ***decomposable*** if there exist β and γ such that $\alpha = \beta + \gamma$. An element which is not decomposable is called ***indecomposable***. The set of all indecomposable elements in L_+ is a base for L. Now we can make the connection to what we learnt above, if we identify the base form L with the base Π of positive simple roots of the Lie algebra $\mathfrak{g}$. Then, the sets L_+ and L_-, as defined in the space E, are exactly the sets of positive and negative roots of the root system of $\mathfrak{g}$ (hence the notation to call them L_+ and L_-).

Geometrically, the reflections, w_α, that generate the Weyl group are reflections with respect to the hyperplane in E perpendicular to the root α. $w_\alpha(\beta)$ would be the vector obtained by reflecting the root β, with respect to the hyperplane perpendicular to α, in E. Each positive simple root α will partition E into two halves such that either $\langle\alpha, \mu\rangle < 0$ or $\langle\alpha, \mu\rangle > 0$. Given Π, we consider the intersection of the sets $\langle\alpha_i, H\rangle > 0$, where α_i are all the elements of Π. This set is called the ***open fundamental Weyl chamber*** in E (relative to Π). The ***closed fundamental Weyl chamber*** in E (relative to Π) is the set of all $H \in E$ such that $\langle\alpha_i, H\rangle \geq 0$ for all $\alpha_i \in \Pi$. One might wonder what do the elements of the closed and open Weyl chambers signify for the Lie algebra $\mathfrak{g}$? These elements are important in the representation theory of semi-simple Lie algebras and are called the ***dominant integral elements***. We will talk about them when we discuss the representation theory of $sl(3, \mathbb{C})$.

The Weyl chamber depends on the base Π and a different, but equivalent, base will give a different Weyl chamber. For each open Weyl chamber C, there exists a unique base Π_C for L such that C is the open fundamental Weyl chamber associated to Π_C and the other way round. So, there is a one-to-one correspondence between Weyl chambers and bases, or set of positive simple roots. The Weyl group acts simply and transitively on the set of positive simple roots and also on the set of Weyl chambers.
We conclude our discussion of the Weyl group with a few properties of $\mathcal{W}$

1. The Weyl group is the set of linear transformations of h^* that leave the set of weights of any representation of $\mathfrak{g}$ invariant.

2. Scalar products are invariant under $\mathcal{W}$.

$$\langle w(\alpha), w(\beta)\rangle = \langle\alpha, \beta\rangle, \tag{3.31}$$

   for any $w \in \mathcal{W}$.

3. The Weyl group acts simply and transitively on the set of positive simple roots and also on the set of Weyl chambers. For any basis Π of simple roots, and for any $w \in \mathcal{W}$, the image $w(\Pi)$ is again a basis of simple roots.

4. The set Π of simple roots generates the whole root system as its image under the Weyl group. For any root α, $\mathcal{W}(\alpha)$ spans the whole root space. This point will be useful later when we reconstruct the BKM Lie superalgebras from their denominator identities.

5. The reflection with respect to a simple root α takes it to its negative, and permutes the rest of the positive roots.

6. $\mathcal{W}$ not only permutes the roots, but the weights of any other highest weight module.

7. Since the Weyl group is generated by the fundamental reflections with respect to the simple roots α_i, any element $w \in \mathcal{W}$ can be written as a 'word' in the fundamental reflections. A given $w \in \mathcal{W}$ may be expressed by different words, and the minimum possible such reflections that generate w is called the length $l(w)$ of w. An expression in the minimum number of reflections is called a ***reduced*** expression.

8. The length l obeys $l(w) = l(w^{-1})$

9. From the definition of the Weyl vector as the sum of all the positive roots, the reflection of ρ with respect to a simple root α_i just takes α_i to $-\alpha_i$ while permuting all the other roots among themselves. Thus, reflection with respect to α_i just subtracts α_i from the Weyl vector.
$$w_{\alpha_i}(\rho) = \rho - \alpha_i \; . \tag{3.32}$$

10. The Weyl vector ρ always lies in the open (and hence closed) Weyl chamber.

11. Each orbit of the Weyl group contains exactly one point in the closed Weyl chamber.

12. The Weyl groups are Coxeter groups. One has
$$\left(w_{\alpha_i} w_{\alpha_j}\right)^{2+|a_{ij}|^2} = 1 \text{ when } |a_{ij}| = 0, 1 \text{ and } i \neq j \; . \tag{3.33}$$
Further, when $|a_{ij}| \geq 2$, there are no relations. The elements a_{ij} are constants related to the roots α_i and α_j.

This concludes our study of the Weyl group for now. We will get back to using it to compute the character of highest weight modules of $\mathfrak{g}$ and its character and denominator identities. It also plays a very important role in constructing the BKM Lie superalgebras from their denominator identities. Next, we come to the idea of classification of finite-dimensional semi-simple Lie algebras and in the process learning about Cartan matrices and Dynkin diagrams.

3.3.6 Cartan Matrices, Dynkin Diagrams and Classification of finite-dimensional semi-simple Lie algebras

Once we have fixed a set, Π, of simple roots of $\mathfrak{g}$, we can associate to it a matrix, of inner products of the positive simple roots $\alpha \in \Pi$, called the ***Cartan matrix***. One can classify the finite-dimensional semi-simple Lie algebras over $\mathbb{C}$ using the Cartan matrix associated to its root system. Enumerating Π as $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, where r is the dimension of E, and hence, of the underlying semi-simple Lie algebra $\mathfrak{g}$, the ***Cartan matrix*** $A(\mathfrak{g})$ of the semi-simple Lie algebra $\mathfrak{g}$ is the $r \times r$ matrix with elements

$$a_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \ . \tag{3.34}$$

The elements a_{ij} are the same ones that appear in the definiton of the Weyl group as a Coxeter group above. Because the positive simple roots form a basis of the root space, the Cartan matrix is non-degenerate, and since the quantity $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ is an integer, all the elements of the Cartan matrix are integers. The Cartan matrix depends on the enumeration of Π and different enumerations lead to different Cartan matrices that are conjugate to one another by a permutation matrix.

To every semi-simple Lie algebra, we can associate a Cartan matrix as defined above. Conversely, a finite-dimensional semi-simple Lie algebra can be defined through its Cartan matrix. Given a real, indecomposable, $(r \times r)$ symmetric matrix[2] $A = (a_{ij})$, $i, j \in I = \{1, 2, \ldots, r\}$ of rank r satisfying the following conditions:

(i) $a_{ij} \in \mathbb{Z}$ for all i and j ,

(ii) $a_{ii} = 2$ for all i ,

(iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$,

(iv) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,

(v) $\det A > 0$,

[2]The symmetric condition can be extended to include symmetrizable matrices. A matrix A is said to be symmetrizable if there exists a non-degenerate diagonal matrix D such that $A = DB$ where B is a symmetric matrix.

one defines a Lie algebra $\mathfrak{g}(A)$ generated by the generators e_i, f_i, and h_i, for $i = \{1, 2, \ldots, r\}$, satisfying the following conditions for $i, j \in I$:

$$\begin{aligned} [h_i, h_j] &= 0 \ ; \ [e_i, f_j] = \delta_{ij} h_j \ ; \\ [h_i, e_j] &= a_{ij} e_j \ ; \ [h_i, f_j] = -a_{ij} f_j \ ; \\ [e_i, e_j] &= [f_i, f_j] = 0 \text{ if } A_{ij} = 0 \\ (\text{ad } e_i)^{1-a_{ij}} e_j &= (\text{ad } f_i)^{1-a_{ij}} f_j = 0 \quad \text{for } i \neq j \ . \end{aligned} \tag{3.35}$$

The above equations may seem a little strange when presented without the necessary motivation, so let us consider what each of the equations above says about the Lie algebra and the roots. (i) says all the entries of the Cartan matrix are integers, which is easy to understand if we note that the quantity $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ is an integer, whose genesis goes back all the way to **(B)**. (ii) is just the choice of a convenient normalization one chooses for the inner product which also goes all the way back to the representations of $sl(2, \mathbb{C})$. Condition (iii) reflects the symmetry of the scalar product in root space. For the meaning of condition (iv), consider the following conditions that one can show on the inner products for any two roots α and β

$$\begin{aligned} (\alpha, \beta) > 0 \qquad &\Rightarrow \qquad \alpha - \beta \in L \\ (\alpha, \beta) < 0 \qquad &\Rightarrow \qquad \alpha + \beta \in L \ . \end{aligned} \tag{3.36}$$

Now, the simple roots were defined as those positive roots which were indecomposable, and hence the difference of any two simple roots is never a root, and so it follows that $(\alpha_i, \alpha_j) \leq 0$, and hence the matrix elements $a_{ij} \leq 0$.

The Cartan matrix, A, uniquely defines the Lie algebra which we call $\mathfrak{g}(A)$. The relations (3.35) are known as the ***Chevalley-Serre*** relations. We had earlier defined general semi-simple Lie algebras as ones which are obtained as direct sums of simple Lie algebras, and also through their Killing form. The Chevalley-Serre relations defining a semi-simple Lie algebra are very useful from the point of view of our final aim of graduating to infinite-dimensional Lie algebras. It is this condition whose generalization is the simplest way to move from finite-dimensional complex semi-simple Lie algebras to infinite-dimensional BKM Lie superalgebras as we will see when we define BKM Lie superalgebras shortly.

Let us look at the entries of the Cartan matrix more closely, and see what we can say about them. First, by the triangle inequality $(\alpha_i, \alpha_j)^2 \leq (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)$, and using (ii) we get the inequality $a_{ij}, a_{ji} \leq 4$ with equality holding for $i = j$ and $a_{ij} \in \{0, 1, 2, 3\}$ for $i \neq j$. Now, using (iii) amd (iv) we see the possibilities for a_{ij} to be

$$\begin{aligned} a_{ij} &= a_{ji} = 0 \quad \text{or} \\ a_{ij} &= a_{ji} = -1 \quad \text{or} \\ a_{ij} &= -1,\ a_{ji} = -2 \quad \text{or} \\ a_{ij} &= -1,\ a_{ji} = -3\ . \end{aligned} \tag{3.37}$$

The elements α_{ij} give the angle between the positive simple roots of the root system. If all the elements of a root system are multiplied by a non-zero constant, one gets another root system that is equivalent to the original root system. The quantity $2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$ is unchanged if both α and β are multiplied by the same non-zero constant. So, the actual lengths of roots are not important, but only their ratios. The elements α_{ij} encode information about the angles between the positive simple roots in the root space E as follows. For two roots α and β, where α is not a multiple of β, and $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$, there are the following possibilities:

(i) $\langle \alpha, \beta \rangle = 0$,

(ii) $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$, and the angle between α and β is 60°or 120°,

(iii) $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$, and the angle between α and β is 45°or 135°,

(iv)
$$\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle, \tag{3.38}$$
and the angle between α and β is 30° or 150° .

So, if the two roots are not multiples of each other and are not perpendicular to each other, then the ratio of their lengths must be either 1, $\sqrt{2}$, or $\sqrt{3}$. If two roots are perpendicular then there are no constraints on the ratios of their lengths. If the angle between two roots α and β is strictly obtuse, then $\alpha + \beta$ is a root, and if the angle between α and β is strictly acute, then $\alpha - \beta$ and $\beta - \alpha$ are also roots. Compare this with the condition (iii) of (3.3.6).

Now we come to the idea of Dynkin diagrams and classification, upto equivalence, of semi-simple Lie algebras. One can classify the root systems, and hence the corresponding semi-simple Lie algebra $\mathfrak{g}$, in terms of the Cartan matrices, or an object called the Dynkin diagram.

To see the motivation, consider the idea of the root space decomposition of the Lie algebra. We consider the maximal abelian subalgebra of $\mathfrak{g}$ and this acts semi-simply on

$\mathfrak{g}$ giving the Lie algebra as a direct sum of root spaces. The Lie algebra is broken up into eigenvectors for the elements of the Cartan subalgebra. Thus, classifying Cartan subalgebras can be extended via the root space decomposition to the classification of Lie algebras. The information of the Cartan subalgebra, and hence of $\mathfrak{g}$, is contained in the set of positive simple roots Π, and hence in the Cartan matrix or Dynkin diagram corresponding to Π. Thus, in classifying the various finite-dimensional semi-simple Lie algebras we are reduced to studying the various classes of Cartan matrices or Dynkin diagrams. We state this in the form of a classification theorem (without proof) towards the end of this section where the various root systems, their corresponding Dynkin diagrams are listed along with the semi-simple Lie algebras they describe. Below we give how one constructs the Dynkin diagram given a root system.

To the set Π or simple roots we can also associate a graph, consisting of vertices and lines connecting them, known as a ***Dynkin diagram***. To each element α_i of Π we associate a vertex v_i. Two vertices v_i and v_j are joined by edges depending upon the angle between the simple roots α_i and α_j. If two roots α_i and α_j are orthogonal then we put no edge between the corresponding vertices v_i and v_j. We put one edge between v_i and v_j if α_i and α_j have the same length, two edges if the longer of α_i and α_j is $\sqrt{2}$ times the shorter, and three edges if the longer of α_i and α_j is $\sqrt{3}$ times the shorter. In addition, if α_i and α_j are not orthogonal or of the same length, we decorate the edge between v_i and v_j with an arrow pointing from the vertex associated to the longer root toward the verted associated to the shorter root. Looking at (3.38) we see that there are only three possible lenght ratios and three possible angles between the roots.

Two Dynkin diagrams are said to be equivalent if there is a one-to-one, and onto map of the vertices of one to the vertices of the other that preserves the number of bonds and the direction of the arrows. Since any two bases Π for the same root system are equivalent because of the action of the Weyl group on them, the equivalence class of Dynkin diagram is independent of the choice of the base Π. Two root systems with equivalent Dynkin diagrams are equivalent. A root system is ***irreducible*** if its Dynkin diagram is connected. We now list all the Dynkin diagrams of the classical semi-simple Lie algebras.

1. A_n : The root system A_n is the root system of the Lie algebra $sl(n+1, \mathbb{C})$. It is of rank n.

2. B_n : The root system B_n is the root system of the Lie algebra $so(2n+1, \mathbb{C})$. It is of rank n.

3. C_n : The root system C_n is the root system of the Lie algebra $sp(n, \mathbb{C})$. It is of rank n.

4. D_n : The root system D_n is the root system of the Lie algebra $so(2n, \mathbb{C})$. It is of rank n.

The Dynkin diagrams associated with the above root systems are given in Fig. (3.1). We note a few interesting points about the above root systems that happen in low rank. In rank one, there is only one possble Dynkin diagram, reflecting that there is only one isomorphism class of complex semi-simple Lie algebras in rank one. The Lie algebra $so(2, \mathbb{C})$ is not semi-simple, and the remaining three Lie algebras $sl(2, \mathbb{C}), so(3, \mathbb{C})$ and $sp(1, \mathbb{C})$ are isomorphic. In rank two, the Dynkin diagram D_2 is disconnected, reflecting the fact that $so(4, \mathbb{C}) \cong sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$. Also, the Dynkin diagrams B_2 and C_2 are isomorphic, reflecting the fact that $so(5, \mathbb{C}) \cong sp(2, \mathbb{C})$. In rank three, the Dynkin diagrams A_3 and D_3 are isomorphic, reflecting the fact that $sl(4, \mathbb{C}) \cong so(6, \mathbb{C})$. In addition to the root systems associated to the classical Lie algebras, there are five exceptional irreducible root systems, denoted G_2, F_4, E_6, E_7 and E_8.

Now we state the classification theorem, without proof below.

Theorem 3.3.6 *Every irreducible root system is isomorphic to precisely one root system from the following list.*

1. *The classical root systems A_n, $n \geq 1$.*

2. *The classical root systems B_n, $n \geq 2$*

3. *The classical root systems C_n, $n \geq 3$*

4. *The classical root systems D_n, $n \geq 4$*

5. *The exceptional root systems G_2, F_4, E_6, E_7 and E_8.*

Since every root system can be uniquely decomposed as a direct sum of irreducible roots sytems, the classification of irreducible root systems leads to the classification of all root systems. As argued before, classification of root systems leads to the classification of semi-simple Lie algebras and classification of the irreducible root systems leads to the classification of the simple Lie algebras which we state in the form of a theorem below.

Theorem 3.3.7 *Every complex simple Lie algebra is isomorphic to precisely one algebra from the following list:*

1. $sl(n+1, \mathbb{C}), \quad n \geq 1,$

2. $so(2n+1, \mathbb{C}), \quad n \geq 2,$

3. $sp(n, \mathbb{C}), \quad n \geq 3,$

4. $so(2n, \mathbb{C}), \quad n \geq 4,$

5. *The exceptional Lie algebras* G_2, F_4, E_6, E_7 *and* E_8.

A semi-simple algebra is a direct sum of simple algebras, and is uniquely determined up to isomorphism by specifying which simple summands occur and how many times each one occurs in the direct sum.

Thus, we have classified the various semi-simple Lie algebras, and their root systems. We have also seen that to each root system one can associate a graph called the Dynkin diagram which pictorially captures all the information about the root system and hence the Lie algebra it corresponds to.

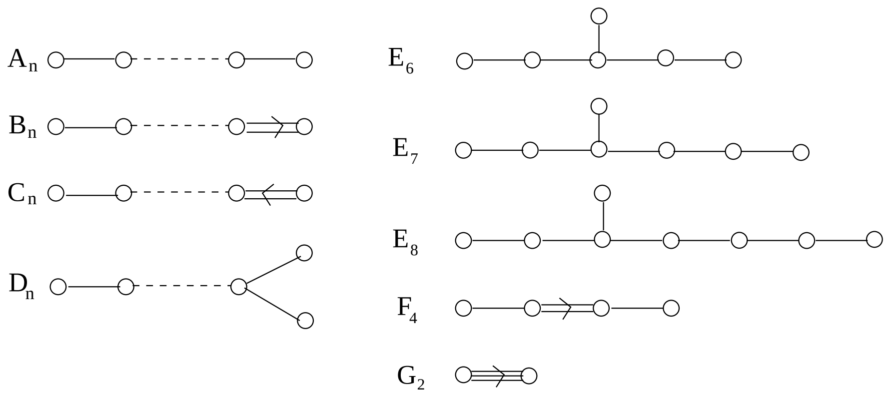

Figure 3.1: The Dynkin diagrams for the classical semi-simple Lie algebras

3.3.7 Representation Theory of Semi-Simple Lie Algebras

After studying the general structure of Lie algebras, we now come to the representation theory of finite-dimensional semi-simple Lie algebras with the example of $sl(3, \mathbb{C})$ as the particular case we work it out explicitly for.

We will still be working with the basis (3.20). Let $\alpha = (a_1, a_2)$ be a root of $sl(3, \mathbb{C})$ and Z_α the corresponding root vector. We have already seen the construction of a representation for $sl(2,\mathbb{C})$. We will use some of what we learnt there and see what modifications occur for Lie algebras of rank $r > 1$.

The starting point in looking for a structure is to find the generalization of **(A)**. We have defined roots and weights as the eigenvalues of $\rho(h)$. The suitable generalization is to observe that in any finite-dimensional representation the Cartan subalgebra $\mathfrak{h}$ acts completely reducibly. The set of operators $\rho(h)$ for all $h \in \mathfrak{h}$ are simultaneously diagonalizable in every finite-dimensional representation, and hence, every finite-dimensional representation (ρ, V) is the direct sum of its weight spaces. The simultaneous eigenvalues of $\rho(h)$ are called *weights*. For $sl(3,\mathbb{C})$, the weights are of the form $\mu = (m_1, m_2)$, where m_1 and m_2 are integers. For a finite-dimensional representation ρ of $\mathfrak{g}$ on a vector space V with Cartan subalgebra $\mathfrak{h}$, an element $\mu \in \mathfrak{h}^*$ is called a ***weight*** for ρ if there exists a non-zero vector v in V such that

$$\rho(h)v_\mu = \langle \mu, h \rangle v_\mu \ , \tag{3.39}$$

for all $h \in \mathfrak{h}$. A non-zero vector v satisfying the above equation is called a ***weight vector*** for the weight μ, and the set of all vectors satisfying (3.39) is called the ***weight space*** with weight μ. The dimension of the weight space is called the ***multiplicity*** of the weight. For any finite-dimensional representation ρ of $\mathfrak{g}$, the weights of ρ and their multiplicity are invariant uinder the action of the Weyl group.

The generalization of **(B)** is the idea of *dominant integral elements*. An ordered pair (m_1, m_2) with m_1 and m_2 being non-negative integers is called a ***dominant integral element*** of $sl(3,\mathbb{C})$. Just like the integers m occured as the highest eigenvalues of the irreducible representations of $sl(2,\mathbb{C})$, we will see that the highest weight of each irreducible representation of $sl(3,\mathbb{C})$ is a dominant integral element and, conversely, that every dominant integral element occurs as the highest weight of some irreducible representation. More generally, for a general semi-simple Lie algebra, an element $\mu \in \mathfrak{h}$ is called an ***integral element*** if $2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ is an integer for each positive simple root α and ***dominant integral*** if it is non-negative. Each weight is an integral element. The set of integral elements is invariant under the action of the Weyl group. It is precisely these elements that are contained in the closed fundamental Weyl chamber. This is the generalization of **(F)** and **(G)** to general semi-simple Lie algebras.

The significance of the roots for the representation theory of semi-simple Lie algebras lies in the generalization of **(E)**. The operators $\rho(x)$ and $\rho(y)$ of $sl(2,\mathbb{C})$ raise and lower, respectively, the eigenvalues of $\rho(h)$. Let $\alpha = (a_1, a_2)$ be a root of $sl(3,\mathbb{C})$ and let Z_α be a corresponding root vector. Let ρ be a representation of $sl(3,\mathbb{C})$, and $\mu(m_1, m_2)$ a weight for

ρ and v the corresponding weight vector. Then,

$$\begin{aligned} \rho(h_1)\rho(z_\alpha)v &= (m_1 + a_1)\rho(Z_\alpha)v \ , \\ \rho(h_2)\rho(z_\alpha)v &= (m_2 + a_2)\rho(Z_\alpha)v \ . \end{aligned} \tag{3.40}$$

Thus, either $\rho(Z_\alpha)v = 0$ or $\rho(Z_\alpha)v$ is a new weight vector with weight

$$\mu + \alpha = (m_1 + a_1, m_2 + a_2) \ . \tag{3.41}$$

For a general semi-simple Lie algebra, let v be a weight vector with weight μ and suppose x_α is an element of $\mathfrak{g}_\alpha$. Then, for all $h \in Fh$ we have

$$\rho(h)\rho(x_\alpha)v = (\langle \mu, h\rangle + \langle \alpha, h\rangle)\rho(x_\alpha)v \ . \tag{3.42}$$

The above equation says that $\rho(x_\alpha)v$ is either zero or is a weight vector with weight $\mu + \alpha$.

For the case of $sl(2,\mathbb{C})$ the weights were integers m and the notion of comparing two weights was just the comparision of the integers, but for a general semi-simple Lie algebra the weight is a collection of the simultaneous eigenvalues of all the $\rho(h_i)$ and we need to clarify what it means to say a weight is *higher* than another. We will illustrate it for the case of $sl(3,\mathbb{C})$. Given the two positive simple roots α_1 and α_2 (eq. (3.23)), and two weights μ_1 and μ_2, we say that μ_1 is ***higher*** than μ_2 (denoted $\mu_1 \succeq \mu_2$) if $\mu_1 - \mu_2$ can be written in the form

$$\mu_1 - \mu_2 = a\alpha_1 + b\alpha_2 \ , \tag{3.43}$$

with $a \geq 0$ and $b \geq 0$. Analogous to the largest eigenvalue in each representation of $sl(2,\mathbb{C})$, there exists a weight μ_0 in each representation of $sl(3,\mathbb{C})$ such that $\mu_0 \succeq \mu$, for all weights μ. This is called the ***highest weight*** of ρ.

Now we have enough ideas to put together the generalizations of **(C), (F)-(I)** to $sl(3,\mathbb{C})$, and any general semi-simple Lie algebra, in the form of a theorem below. In the case of $sl(2,\mathbb{C})$ each irreducible representation $\rho(h)$ is diagonalizable, and there is a largest eigenvalue of $\rho(h)$. The essence of **(C), (F)-(I)** is that any two irreducible representations of $sl(2,\mathbb{C})$ with the same largest eigenvalue are equivalent. The highest eigenvalue is always a non-negative integer, and, conversely, for every non-negative integer m, there is an irreducible representation with hightest eigenvalue m. Now we state the theorem of highest weight for $sl(3,\mathbb{C})$ and generalize it to any finite-dimensional semi-simple Lie algebra.

Theorem 3.3.8 *1. Every irreducible representation ρ of $sl(3,\mathbb{C})$ is the direct sum of its weight spaces; that is, $\rho(h_1)$ and $\rho(h_2)$ are simultaneously diagonalizable in every irreducible representation. More generally, in every finite-dimensional representation irreducible representation (ρ, V) is the direct sum of its weight spaces.*

2. *All the weights, μ, are integral elements.*

3. *Every irreducible representation of $sl(3,\mathbb{C})$ has a unique highest weight μ_0, and two equivalent irreducible representations have the same highest weight. And any two irreducible reesentations of $sl(3,\mathbb{C})$ with the same highest weight are equivalent. The same is true for any general semi-simple Lie algebra.*

4. *Two irreducible representations with the same highest weight are equivalent.*

5. *If π is an irreducible representation of $sl(3,\mathbb{C})$, then the highest weight μ_0 of π is of the form*

$$\mu_0 = (m_1, m_2)$$

with m_1 and m_2 being non-negative integers. The suitable generalization is the statement that the highest weight of every irreducible representation is a dominant integral element.

6. *If m_1 and m_2 are non-negative integers, then there exists an irreducible representation ρ of $sl(3,\mathbb{C})$ with highest weight $\mu_0 = (m_1, m_2)$. For a general semi-simple Lie algebra, every dominant integral element occurs as the highest weight of an irreducible representation.*

The trivial representation is an irreducible representation with highest weight $(0,0)$. For $sl(2,\mathbb{C})$ an irreducible representation with highest weight m was of dimension $(m+1)$. For $sl(3,\mathbb{C})$, the dimension of the irreducible representation with highest weight (m_1, m_2) is

$$\tfrac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2) \ .$$

We will come back to the representation theory of finite-dimensional Lie algebras briefly towards the end of the chapter when we discuss the denominator identity of Lie algebras. For now, we just recapitulate what we have learnt about finite-dimensional Lie algebras, before we move on to the topic of infinite-dimensional Lie algebras.

(**a**) Generalizing **(A)** we see that there exists a maximal abelian subalgebra, called the Cartan subalgebra, of $\mathfrak{g}$ which acts semi-simply on $\mathfrak{g}$ and every irreducible representation is given as the direct sum of weight spaces with respect to the Cartan subalgebra. The eigenvalues are the roots and weights of the Lie algebra.

(**b**) The multiplicity of every root is one.

(**c**) Generalizing **(B)**, all the weights are integral elements.

(**d**) Every irreducible representation of $\mathfrak{g}$ has a unique highest weight which is a dominant integral element and two equivalent irreducible representations have the same highest weight. See point **(C), (H), (I)**

(**e**) Every dominant integral element is the highest weight of an irreducible representations. See **(G)**

(**f**) The set of roots can be divided into positive and negative roots with respect to a basis of positive simple roots. The choice of simple roots is not unique, nor is their ordering.

(**g**) The Lie algebra splits as a direct sum

$$\mathfrak{g} = \bigoplus_{\alpha \in L} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in L_-} \mathfrak{g}_\alpha \bigoplus \mathfrak{h} \bigoplus_{\alpha \in L_+} \mathfrak{g}_\alpha, \tag{3.44}$$

and all the root spaces are one-dimensional.

(**h**) There is a group of permutations of the set of positive simple roots, which is generated by reflections with respect to the set of positive roots in the root space, known as the Weyl group. The set of fundamental refelctions generate the Weyl group.

(**i**) The Weyl group is finite-dimensional. It breaks up the root space into chambers known as the Weyl chambers.

(**j**) There exists a vector ρ called the **Weyl vector** which always lies in the closed Weyl chamber.

(**k**) The set of positive simple roots capture all the information of the Lie algebra $\mathfrak{g}$. The inner product matrix constructed from the inner products of the various positive simple roots in the root space is called the Cartan matrix. It contains all the information about the Lie algebra $\mathfrak{g}$.

(l) Another equivalent description of the Lie algbra $\mathfrak{g}$ is through its Dynkin diagram which contains the same information as the Cartan matrix.

(m) The semi-simple Lie algebra $\mathfrak{g}$ can be described equivalently through its bilinear form or through its Chevalley-Serre relations.

(n) There are four classes of classical root systems, namely, A_n $(n \geq 1), B_n$ $(n \geq 2), C_n$ $(n \geq 3)$, and D_n $(n \geq 4)$ and five exceptional root systems, namely, G_2, F_4, E_6, E_7 and E_8. Every irreducible root system is isomorphic precisely one root system from the above list.

(o) Every simple Lie algebra is isomorphic precisely to one algebra from amongst $sl(n+1,\mathbb{C}),\quad n \geq 1$, $so(2n+1,\mathbb{C}),\quad n \geq 2$, $sp(n,\mathbb{C}),\quad n \geq 3$, $so(2n,\mathbb{C}),\quad n \geq 4$, and the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8.

(p) There is a one-to-one correspondence between the classes of simple Lie algebras and root systems.

3.4 Infinite Dimensional Lie Algebras

We start with the theory of infinite-dimensional Lie algebras now. One thing working for us is we know which directions to look in and what roughly to expect in pursuing them. To explore the structure, we should start by trying to find a Cartan subalgebra, (or whatever the generalization of that could be in the infinite-dimensional case) and find simultaneous eigenvectors and the corresponding eigenvalues of all its elements. This should give us the root structure, and root space decomposition of the Lie algebra. Then, we find a base of positive simple roots for the set of roots, and from this construct the Cartan matrix and, if it exists, the Dynkin diagram corresponding to the Lie algebra. There will be the generalization of the Weyl group, character and denominator formulae also as we will see.

Before we proceed with the study of BKM Lie superalgebras, we should mention that although BKM Lie superalgebras are generalizations of finite-dimensional Lie algebras, this simplistic way of studying them is useful only to overcome the initial bridge in intuition necessary to appreciate their abstract theory, but one should let go of the crutch at the earliest to completely appreciate the theory of BKM Lie superalgebras by itself. For one thing, the various other branches like vertex algebras, vector valued modular forms, etc.

play a rich role in the theory of BKM Lie superalgebras which gives it a lot of structure than a mere extrapolation from finite-dimensional Lie algebras would suggest.

Sadly, the scope of the introduction we will give, given the space available to introduce it, will rely heavily on borrowing intuition from the finite-dimensional case to seek motivation and justification of the various constructs in BKM Lie superalgebras. We will use examples to bridge the gaps in intuition.

It is interesting to go back to the origin of finite-dimensional Lie algebras and note that they were originally constructed to study Lie groups, while for the case of infinite-dimensional Lie algebras it was the Lie algebras that were constructed first, and for the case of BKM Lie superalgebras the corresponding group structure is far from being fully understood. As the degree of generalization increases, the group structure becomes less clear.

Ordering them in an increasing sequence of complexity, one obtains affine Lie algebras as the simplest generalization of finite-dimensional Lie algebras by central extension of loop algebras. The center of a finite-dimensional semi-simple Lie algebra is trivial. One can form a first generalization by constructing what is known as the 'loop algebra' of a finite-dimensional semi-simple Lie algebra. To make it consistent one needs to add a derivation to its center and this algebra is the corresponding affine Lie algebra. Affine Lie algebras are a sub-class of the class of infinite-dimensional Lie algebras known as Kac-Moody algebras. Borcherds-Kac-Moody Lie algebras were constructed by Borcherds as a generalization of Kac-Moody Lie algebras and are the most general class of Lie algebras.

3.4.1 Loop Algebras and Central Extensions

We start our study of infinite-dimensional Lie algebras with the simplest class of infinite-dimensional Lie algebras, namely ***affine Lie algebras***. The general construction of affine Lie algebras is along the lines we will describe for general BKM Lie superalgebras, but here we study them in a way that illustrates the transition from finite-dimensional Lie algebras to their infinite-dimensional couterparts. We will construct them as loop algebras of finite-dimensional Lie algebras.

We will describe here a first example of an infinite-dimensional Lie algebra, that of affine Lie algebras as central extensions of loop algebras. The advantage of this construction is that it is realized entirely in terms of an underlying simple finite-dimensional Lie algebra, known as its derived algebra.

The center of a finite-dimensional semi-simple Lie algebra is trivial. The existence of a central element, as we will see, is a feature that we will find in all infinite-dimensional Lie algebras. Given a finite-dimensional Lie algebra $\mathfrak{g}$, we can try and construct an (l-dimensional) central extension to it by simply adding l central generators, k_i, to the algebra and imposing

$$[t^\alpha, k^i] = 0 \quad \text{for} \quad i = 1, \ldots, l, \quad \alpha = 1, \ldots, r\ . \tag{3.45}$$

This will modify the brackets between the original generators to include the central generators as follows

$$[t^\alpha, t^\beta] = f^{\alpha\beta}_{\ \ \gamma}\, t^\gamma + f^{\alpha\beta}_{\ \ i}\, k^i, \tag{3.46}$$

where $f^{\alpha\beta}_{\gamma}$ are the structure constans of $\mathfrak{g}$. The new structure constants $f^{\alpha\beta}_{\ \ i}$ have to satisfy the Jacobi identity and thus cannot be completly arbitrary. The number of solutions to the above equation subject to the Jacobi identity constraint is the number of independent central extensions one can write down for $\mathfrak{g}$. Finding the k^i from the above equation shows, for finite-dimensional Lie algebras, that the trivial solution $f^{\alpha\beta}_{\ \ i} = 0$ is the only possible solution. Hence, the center of a finite-dimensional semi-simple Lie algebra is trivial. Thus, to centrally extend the Lie algebra $\mathfrak{g}$, we need to alter its structure to allow for the extension. This leads to the idea of the loop algebra of a finite-dimensional Lie algebra.

Let $\mathfrak{g}$ be a simple Lie algebra, and consider the space of analytic maps from the circle S^1 to $\mathfrak{g}$. As before, let $\{t^\alpha | \alpha = 1, \ldots, r\}$ be a basis of $\mathfrak{g}$, and S^1 be the unit circle in the complex plane with coordinate z. Then a basis for the above vector space of analytic maps from S^1 to $\mathfrak{g}$ will be of the form $\{t^\alpha_n | \alpha = 1, \ldots, r;\ n \in \mathbb{Z}\}$, where $t^\alpha_n = t^\alpha \otimes z^n$. This space inherits a natural bracket operation from the Lie algebra $\mathfrak{g}$ as

$$[t^\alpha_m, t^\beta_n] \equiv [t^\alpha \otimes z^m, t^\beta \otimes z^n] = [t^\alpha, t^\beta] \otimes (z^m \cdot z^n), \tag{3.47}$$

and thus,

$$[t^\alpha_m, t^\beta_n] = f^{\alpha\beta}_{\ \ \gamma}\, t^\gamma \otimes z^{m+n} = f^{\alpha\beta}_{\ \ \gamma}\, t^\gamma_{m+n}, \tag{3.48}$$

where $f^{\alpha\beta}_{\ \ \gamma}$ are structure constants of $\mathfrak{g}$. With the above bracket this space becomes a Lie algebra called the ***loop algebra***, denoted $\mathfrak{g}_{loop}$. Note that the subalgebra of $\mathfrak{g}_{loop}$ generated by the generators t^α_0 is just the subalgebra $\mathfrak{g}$. Note that, the algebra $\mathfrak{g}_{loop}$ has an ***infinite*** number of generators.

Now, we can look for a central extension to $\mathfrak{g}_{loop}$ in the same way as we did for $\mathfrak{g}$. We try the most general ansatz for the bracket of the generators as

$$[t^\alpha_m, t^\beta_n] = f^{\alpha\beta}_{\ \ \gamma}\, t^\gamma \otimes z^{m+n} + (f^{\alpha\beta}_{\ \ i})_{mn} k^i \ . \tag{3.49}$$

We impose the constraints coming from the Jacobi identity and the fact that the algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}_{loop}$, and hence the structure constants $(f^{\alpha\beta}_{\ \ i})_{00}$ and $(f^{\alpha\beta}_{\ \ i})_{m0}$ can be put to zero. Now, for a fixed value of n, the generators t^α_n transform just like the generators t^α (that is, in the adjoint) and hence the structure constants should from an invariant tensor of the adjoint representation of $\mathfrak{g}$ with respect to the indices α, β. It turns out there is a unique such tensor for $\mathfrak{g}$, and that is the Killing form of $\mathfrak{g}$. Thus, the central extension is only one-dimensional, and is proportional to the Killing form B of $\mathfrak{g}$. For convenience, we can choose a basis such that the Killing form in that basis is equal to $\delta^{\alpha\beta}$. This gives the following brackets for $\mathfrak{g}_{loop}$

$$[t^\alpha_m, t^\beta_n] = f^{\alpha\beta}_{\ \ \gamma}\, t^\gamma \otimes z^{m+n} - \ mk\delta^{\alpha\beta}\delta_{m+n,0},$$

$$[k, t^\alpha_n] = 0 \ . \tag{3.50}$$

The infinite-dimensional loop algebra, is usually written in the following way. Let $\mathcal{L} = \mathbb{C}[z, z^{-1}]$ be the algebra of Laurent polynomials in z. Then the loop algebra is then given by

$$\mathfrak{g}_{loop} = \mathcal{L} \otimes_{\mathbb{C}} \mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}. \tag{3.51}$$

We need to add one more generator, d, to this centrally extended infinite-dimensional algebra, known as the derivation which has the following brackets with the other generators [65, 66]

$$[d, t^\alpha_m] = -[t^\alpha, d] = mt^\alpha_m; \quad [d, k] = 0 \ . \tag{3.52}$$

The above construction is then

$$\widehat{\mathfrak{g}} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}. \oplus \mathbb{C}k \oplus \mathbb{C}d \ . \tag{3.53}$$

The generators t^α_0 have vanishing brackets with the derivation, and the subalgebra generated by them is just the Lie algebra $\mathfrak{g}$ known as the ***horizontal subalgebra*** of the affine Lie algebra $\widehat{\mathfrak{g}}$.

3.4.2 The Root System

We first determine the maximal abelian subalgebra. This will certainly contain the Cartan subalgebra of the horizontal subalgebra $\mathfrak{g}$, generated by $h_i \in \mathcal{H}$ $i = 1, \ldots, r$. It will also contain the central element k and d. Thus, the Cartan subalgebra is

$$\widehat{\mathcal{H}} = \text{span}\{k, d, h_0^i| \;\; i = 1, \ldots, r\} \tag{3.54}$$

The roots with respect to $\widehat{\mathcal{H}}$ can be found by observing the following relations

$$[h_0^i, e_n^j] = (h_0^i, h_j^0)e_n^j, \quad [k, e_n^j] = 0, [d, e_n^j] = ne_n^j \tag{3.55}$$

and

$$[h_0^i, h_n^j] = [k, h_n^j] = 0, \quad [d, h_n^j] = nh_n^j \;. \tag{3.56}$$

Writing the roots $\hat{\alpha}$ suggestively as a triplet of eigenvalues under the generators (h_0^i, k, d), the set of roots is

$$\hat{\alpha}_n^i = (\alpha^i, 0, n), \quad \alpha \in L(\mathfrak{g}), n \in \mathbb{Z}, \tag{3.57}$$

and

$$\hat{\alpha}_n^0 = (0, 0, n), \quad n \in \mathbb{Z}\backslash\{0\}, \tag{3.58}$$

corresponding to the generators e_n^i and $h_n^j, n \neq 0$, respectively. The root α^i is a root of the horizontal subalgebra $\mathfrak{g}$. Each of the roots $\hat{\alpha}_n^i$ of (3.57) has multiplicity one, while the root $\widehat{\alpha}_n^0$ has multiplicity r, since it does not depend on the label j of the h_0^j generators and is an eigenvalue of each h_0^j with the eigenvalue 0 and there are r such generators.

Here we can understand the necessity to include the generator d in the Cartan subalgebra. Without the generator d to distinguish the level n of the root, all the roots are infinitely degenerate. The generator d thus 'grades' the algebra $\widehat{\mathfrak{g}}$ according to the level n and only then are the roots $\widehat{\alpha}_n^i$ of (3.57) are non-degenerate. We cannot, however, remove the degeneracy of the root $\widehat{\alpha}_n^0$ in (3.58).

The set of roots of $\widehat{\mathfrak{g}}$ is denoted by $\widehat{L}$, and the root system of the horizontal subalgebra $\mathfrak{g}$ is just the subset $\widehat{\alpha}_0^i = (\alpha^i, 0, 0)$, denoted L as before.

So far the root systems of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ are very much similar and constructed along the same lines. The roots are just the collection of simultaneous eigenvalues of the elements of the Cartan subalgebra. However, the Cartan subalgebra in this case is centrally extended to include two more generators and the roots also contain the eigenvalues of these generators.

All the simple roots have multiplicity one as before. There is, however, one major difference coming from the infinite-dimensional nature of the algebra. It is the appearance of a root $\widehat{\alpha}_n^0$ with multiplicity r.

Like with the finite-dimensional semi-simple case, we find a set of simple roots, and decompose the set of roots into positive and negative roots, with respect to the set of positive simple roots. We identify the set of positive simple roots as

$$\widehat{\alpha}_0^i = (\alpha^i, 0, 0) = \alpha^i \quad \text{for} \quad i = 1, \ldots, r, \tag{3.59}$$

and

$$\widehat{\alpha}_1^0 = (-\mu, 0, i) = \delta - \mu \ . \tag{3.60}$$

Here μ is the highest root of $\widehat{\mathfrak{g}}$ and $\delta = (0,0,1)$. With this, the degenerate roots (3.58) are just $\widehat{\alpha}_n^0 = n \cdot \delta, n \neq 0$. The root $\widehat{\alpha}_1^0$ is called an ***imaginary*** root. It is **not** a simple root in the sense that it is decomposable. Later we will see, in the context of BKM Lie superalgebras imaginary roots that are also simple. However, we include $\widehat{\alpha}_n^0$ in our basis of simple roots. With this identification of positive simple roots, the set of positive roots is

$$\widehat{L}_+ = \{\widehat{\alpha} = (\alpha, 0, n) \in \widehat{L} | \ \ n > 0 \ \text{ or } \ (n = 0, \alpha \in L)\}, \tag{3.61}$$

and the set of negative roots is $\widehat{L}_- = \widehat{L} \backslash \widehat{L}_+$. Denoting the subalgebras generated by the positive and negative roots by $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ respectively, we again have a triangular decomposition of the Lie algebra $\widehat{\mathfrak{g}}$ as

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \widehat{\mathcal{H}} \oplus \widehat{\mathfrak{g}}_- \ . \tag{3.62}$$

3.4.3 Weyl Group

In analogy with simple Lie algebras, one defines the ***Weyl group*** of reflections of the weight lattice of an affine Lie algebra. First, we define a reflection as follows.

$$w_\alpha \cdot \beta = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \ . \tag{3.63}$$

Note that all the α above are the real roots, because the denominator of the RHS would not make sense for an imaginary root. Because of the above form of the reflection, many properties of the affine Weyl group are analogous to those of the Weyl group of simple Lie algebras. There are, however, also new features which are relatied to the existence of imaginary roots. In particular, note that $\langle \alpha, \delta \rangle = 0$ for any real root α, and hence one has

$$w_\alpha \cdot \beta = \delta, \tag{3.64}$$

and hence any Weyl reflection acts as the identity on the set $\widehat{L}_{im} = \{n\delta | \ n \neq 0\}$ of imaginary roots,

$$w_\alpha | \ \widehat{L}_{im} = id_{\widehat{L}_{im}} \ . \tag{3.65}$$

Since any reflection is an automorphism of the root lattice, this also means that the Weyl group maps the set $\widehat{L}_r$ of real roots onto itself.

$\widehat{\mathcal{W}}$ is the semidirect product of the Weyl group of $\mathfrak{g}$ and the group of translations in the coroot lattice.

$$\widehat{\mathcal{W}} = \mathcal{W} \ltimes T \ . \tag{3.66}$$

The roots, however, now have additional eigenvalues in them and this would show up in the various computations of the Weyl group, and we will see that now. Let $\widehat{\alpha}^i_n = (\alpha^i, 0, n)$ and $\widehat{\beta}^i_m = (\beta^i, 0, m)$ be two real roots. Then, the reflection $w_{\widehat{\alpha}^i_n} \cdot \widehat{\beta}^i_m$ is given by

$$w_{\widehat{\alpha}^i_n} \cdot \widehat{\beta}^j_m = \widehat{\beta}^j_m - \frac{2}{\langle \widehat{\alpha}^i_n, \widehat{\alpha}^i_n \rangle} [\langle \alpha^i, \beta^j \rangle + 0 \cdot m + n \cdot 0] \widehat{\alpha}^i_n \ . \tag{3.67}$$

The reflection can be expressed, again, as a triplet like the roots $\widehat{\alpha}^i_n$ as (here we consider a general weight $\widehat{\mu}^j_m = (\mu^j, k, m)$, and denote the quantity $\frac{2}{\langle \widehat{\alpha}^i_n, \widehat{\alpha}^i_n \rangle}$ by $\alpha^{i\vee}$).

$$w_{\widehat{\alpha}^i_n} \cdot \widehat{\mu}^j_m = \Big(w_{\alpha^i} \cdot (\mu^j + nk\alpha^{i\vee}), k, m + \frac{1}{2k} [\langle \mu^j, \mu^j \rangle - \langle \mu^j + nk\alpha^{i\vee}, \mu^j + nk\alpha^{i\vee} \rangle] \Big) \ . \tag{3.68}$$

This is the expression for the reflection of a weight $\widehat{\mu}^j_m$ with respect to a real root $\widehat{\alpha}^i_n$. We can get a very intuitive picture of the structure of the Weyl group if we carry out one more computation that allows us to recast the above reflection in a very suggestive form. Defining for any root $\beta^j \in L$, the translation T^i_α as

$$T^i_\alpha : \widehat{\mu}^j_m = (\mu^j, k, m) \mapsto \Big(\mu^j + k\alpha^i, k, m + \frac{1}{2k} [\langle \mu^j, \mu^j \rangle - \langle \mu^j + k\alpha^i, \mu^j + k\alpha^i \rangle] \Big) \ . \tag{3.69}$$

Using this we can write (3.68) as

$$w_{\widehat{\alpha}^i_n} = w^i_\alpha \circ (T_{\alpha^{i\vee}})^n, \tag{3.70}$$

where w^i_α is the ordinary Weyl reflection which acts on the first component of the triplet of the root μ^i_m and as the identity on the last two components. Thus, we see that any Weyl reflection of the affine Weyl group can be written of the form

$$w_{\widehat{\alpha}^i_n} = w^i_\alpha \circ T^j_\beta \tag{3.71}$$

for some β^j. Also $T_{w^i_\alpha} = w \circ T_{w^i_\alpha} \circ w^{-1}$ for all $w \in \widehat{\mathcal{W}}$. The abelian group of translations is a normal subgroup of $\widehat{\mathcal{W}}$ with $\mathcal{W} \cap T = \{\mathrm{id}\}$. Thus, the affine Weyl group $\widehat{W}$ is a semidirect product of the Weyl group of the horizontal subalgebra and a group of translations T,

$$\widehat{\mathcal{W}} = \mathcal{W} \ltimes T \ . \tag{3.72}$$

An important property of the translations group is that it is generated by the highest root appearing in the definition of the zero root in (3.58)as

$$w_{\widehat{\alpha}^0_1} \cdot \mu = (w_\theta \cdot \mu^i + k\theta^\vee, k, m + \langle \mu^i, \theta^\vee \rangle - k^\vee) \ . \tag{3.73}$$

As a consequence, all possible translations are obtained by reflection with respect to the root α^0_1, and combinations of this reflection with elements of $\mathcal{W}$. Thus, we see that after recasting the affine Weyl reflection in the above form the Weyl group of the horizontal subalgebra generates the reflections, while the imaginary root generates translations. Thus, the affine Weyl group is generated by

$$w_i \equiv w_{\widehat{\alpha}^i}, \quad i = 1, \ldots, r \ . \tag{3.74}$$

The affine Weyl group is infinite-dimensional as against the finite-dimensional Weyl group $\mathcal{W}$ which is finite-dimensional. The affine Weyl group also permutes transitively and freely the affine Weyl chambers which are those open subsets of the weight space which are obtained by removing all the hyperplanes which are left invariant by some Weyl reflection. Similarly, we also define the ***dominant affine Weyl chamber***

$$P^+_k \Big\{ \sum_{i=0}^{r} \mu^i \mu_i | \ \ \mu^i \geq 0 \Big\} \ . \tag{3.75}$$

The algebra also admits a Weyl vetor which is defined as follows:

Definition 3.4.1 *A Weyl vector is defined to be a vector ρ satisfying*

$$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i), \tag{3.76}$$

for all real simple roots α_i. We will see that the above definition also extends to the more general infinite-dimensional Lie algebras.

3.4.4 Classification of Affine Lie Algebras

Just like for the case of finite-dimensional semi-simple Lie algebras, one can also classify the various classes of affine Lie algebras via their root systems and equivalently through their Dynkin diagrams[67, 68]. There are four infinite classes of root systems called A_r, B_r, C_r and D_r. In addition, there are five exceptional affine Lie algebras called E_6, E_7, E_8, F_4 and G_2. Below, we list the Dynkin diagrams for these classes of affine Lie algebras.

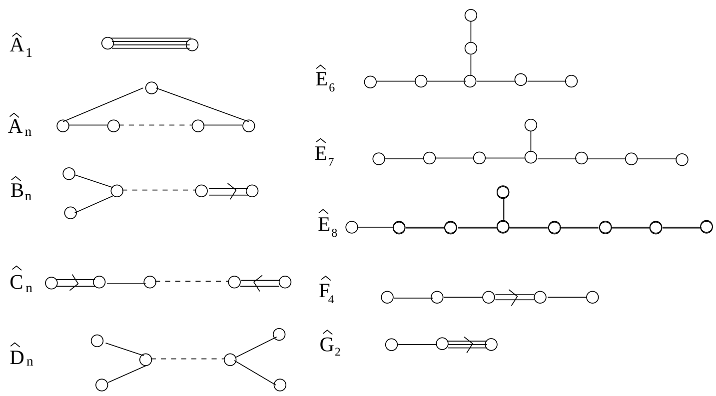

Figure 3.2: The Dynkin diagrams for the affine Lie algebras

This concludes our discussion of affine Lie algebras. We will come back to them later when we discuss the Weyl denominator formula for Lie algebras. We will now give a brief introduction to the theory of super-algebras before going to discuss BKM Lie superalgebras.

3.5 A Brief Introduction to BKM Lie superalgebras

3.5.1 Superalgebras

Let us start with the notion of a super vector space. A ***super vector space*** is a vector space over a field that is $\mathbb{Z}_2$-graded, i.e. has the decomposition

$$V = V_0 \oplus V_1, \quad 0,1 \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \ , \tag{3.77}$$

and in general an M-graded vector space has the decomposition

$$V = \bigoplus_{\alpha \in M} V_\alpha \ . \tag{3.78}$$

An element of V_α is said to be ***homogeneous*** of degree α. For a vector space V, its tensor algebra $T(V)$, symmetric algebra $S(V)$, and the exterior algebra $\bigwedge V$ are examples of graded vector spaces. A ***superalgebra*** is a $\mathbb{Z}_2$-graded algebra, $A = A_0 \oplus A_1$, for which $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. The grading can be more general as above for vector spaces, but we will mainly be considering $\mathbb{Z}_2$-gradings.

A sub-algebra of a superalgebra is also a superalgebra, and a sub-super algebra I of $\mathfrak{g}$ is called an ideal if $[I, \mathfrak{g}] \subseteq I$. A Lie superalgebra is defined similar to a regular Lie algebra, but now one has to keep in mind the consistency imposed by the grading.

Definition 3.5.1 *A Lie superalgebra is a $\mathbb{Z}_2$ graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a Lie bracket satisfying*

$$[x, y] = -(-1)^{deg(x)deg(y)}[y, x] \tag{3.79}$$

and

$$[x[y, z]] = [[x, y]z] + (-1)^{d(x)d(y)}[[x, z]y] \ , \tag{3.80}$$

where for any homogeneous element $g \in \mathfrak{g}_n, n = 0, 1, deg(g) = n$. The subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are called the even and odd parts of $\mathfrak{g}$.

A Lie superalgebra is not a Lie algebra the way one understands semi-simple Lie algebras. $\mathfrak{g}_0$ is an ordinary Lie algebra, while $\mathfrak{g}_1$ is a $\mathfrak{g}_0$ module. Consider the associative algebra endomorphisms $gl(V)$ of the supervector space V. It has a natural $\mathbb{Z}_2$ grading as follows

$$\begin{aligned} gl(V)_0 &= \{f \in gl(V) : f(V_n) \subseteq V_n, \quad n \in \mathbb{Z}_2\} \ , \\ gl(V)_1 &= \{f \in gl(V) : f(V_n) \subseteq V_{n+1}, \quad n \in \mathbb{Z}_2\} \ . \end{aligned} \tag{3.81}$$

The Lie bracket is defined as follows

$$[x,y] = \begin{cases} xy - yx, & \text{if } x \text{ or } y \in gl(V)_0 \ , \\ xy_yx, & \text{if } x, y \in gl(V)_1 \ . \end{cases} \tag{3.82}$$

A good reference for Lie superalgebras is [69, 70]

3.5.2 BKM Lie superalgebras

Due to constraints of space and scope, our introduction of BKM Lie superalgebras will be top-down (mostly following [55]). The milestone approach by which we studied finite-dimensional Lie algebras is not possible here and the only way one can actually understand or apperciate the subject is by undertaking a detailed study of it. Short of that, we use the ideas already constructed in the context of finite-dimensional semi-simple Lie algebras and affine Lie algebras to motivate and make the results in this section look plausible.

Our main interest in BKM Lie superalgebras from the point of view of the problem of counting dyons is in the Weyl-Kac-Borcherds (WKB) denominator formula. Our introduction of BKM Lie superalgebras is given with the very narrow aim of understanding the WKB denominator formula. We will learn only so much as will allow us to state and understand the denominator identity. For more on the subject the reader is refered to the literature on the subject.

In what follows we will use the following notation. We will use $\mathcal{G}$ to denote a BKM Lie superalgebra. We will use the set I to index the set of generators of the BKM Lie superalgebra. It will either be the set $\{1, \ldots, n\}$ or a countably infinite set in which case it is identified with $\mathbb{N}$. We will use the set $S \subseteq I$ to index the odd generators. We continue using e_i, h_i, and f_i for the generators of the Lie algebra. The Cartan subalgebra of $\mathcal{G}$ will be denoted $\mathcal{H}$.

We will first define BKM Lie superalgebras through their Chevalley-Serre relations. The advantage of this is that it gives us an understanding of the structure of the BKM Lie superalgebra in terms of the generators right at the very beginning. We will pursue alternate characterizations to augment this point of view later on. We have already seen the development and definition of the Chevalley-Serre relations for the finite-dimensional case. The key to the definition was the root space decomposition of the Lie algebras made possible by the semi-simple action[3] of the Cartan subalgebra on the Lie algebra. To be able to carry out the same procedure for infinite-dimensional Lie algebras, we first need to define the abelian Lie algebra that will be the Cartan subalgebra which we do now.

Let $\mathcal{H}_{\mathbb{R}}$ be a real vector space with a non-degenerate symmetric real valued bilinear form $(.,.)$ and elements $h_i, i \in I.$ such that

(i) $(h_i, h_j) \leq 0$ if $i \neq j$,

(ii) If $(h_i, h_i) > 0$, then $\frac{2(h_i,h_j)}{(h_i,h_i)} \in \mathbb{Z}$ for all $j \in I$,

(iii) If $(h_i, h_i) > 0$ and $\imath \in S$, then $\frac{(h_i,h_j)}{(h_i,h_i)} \in \mathbb{Z}$ for all $j \in I$.

Let $\mathcal{H} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The above definition seems too ad hoc, unmotivated and uncomfortable to accept, so let us try to convince ourselves that it indeed has the properties we have come to expect a Cartan sub-algebra to have from our study of finite-dimensional semi-simple Lie algebras and affine Lie algebras. Comparing (i) above with the definition 3.3.3 we see that with the addition of the requirement that inner product can be less than 0, (i) essentially is a generalization of 3.3.3. Comparing (ii) and (iii) above with the conditions on the Cartan matrix in section 3.3.6 and eq. (3.34) to see where the motivations and generalizations come from. Having defined the Cartan subalgebra, we can now define a BKM Lie superalgebra via its action on it. These will be the Chevalley-Serre relations for the BKM Lie superalgebra.

Before looking at the Chevalley-Serre relations for the BKM Lie superalgebra, recall the relations (3.9) and (3.35). With those in mind we now define the following:

Definition 3.5.2 *A Borcherds-Kac-Moody Lie superalgebra $\mathcal{G} = (A, \mathcal{H}, S)$ associated to the Cartan matrix A, with the abelian Lie algebra $\mathcal{H}$ as its Cartan subalgebra, is the Lie superalgebra generated by $h_i \in \mathcal{H}$ and elements e_i, f_i with $i \in I$ satisfying the following defining relations:*

(i) $[e_i, f_j] = \delta_{ij} h_i$

(ii) $[h, e_i] = (h, h_i)e_i,\ [h, f_j] = -(h, h_j)f_j,$

[3]A linear operator on a finite-dimensional vector space is said to act *semi-simply* if the complement of every invariant subspace of the operator is also an invariant subspace. An important result for such a linear operators on a finite-dimensional vector space over an algebraically closed field is that it is diagonalizable.

(iii) $deg\ e_i = 0 = deg\ f_i$ *if* $i \notin S$, $deg\ e_i = 1 = deg\ f_i$ *if* $i \in S$,

(iv) $(ad\ e_i)^{1-\frac{2a_{ij}}{a_{ii}}} e_j = (ad\ f_i)^{1-\frac{2a_{ij}}{a_{ii}}} f_j = 0 \quad$ *if* $a_{ii} > 0$ *and* $i \neq j$.

(v) $(ad\ e_i)^{1-\frac{a_{ij}}{a_{ii}}} e_j = (ad\ f_i)^{1-\frac{a_{ij}}{a_{ii}}} f_j = 0 \quad$ *if* $i \in S, a_{ii} > 0$ *and* $i \neq j$.

(vi) $[e_i, e_j] = 0 = [f_i, f_j]$ *if* $a_{ij} = 0$.

Let us understand the above definition. Looking at (3.35) we can understand the origin of the relations $(i), (ii), (iv)$ and (v) above. The condition (iii) is expected of a super Lie algebra where we will need to distinguish between the even and odd elements of the Lie algebra (see also (3.81)). As expected, when $S = \emptyset$, we have a Lie algebra and condition (iii) is redundant. The center of the BKM Lie superalgebra is contained in the Cartan subalgabra $\mathcal{H}$. Recall that finite-dimensional semi-simple Lie algebras have a trivial center, while in constructing affine Lie algebras we had to add a central extension and a derivation to the center to make the algebra consistent and hence the center was not trivial. BKM Lie superalgebras also have a non-trivial center. The Cartan subalgebra $\mathcal{H}$ acts semisimply on the BKM Lie superalgebra $\mathcal{G}$ via the adjoint actions (which was the whole point of constructing it).

The matrix A, we recognize, is the generalized symmetric Cartan matrix of the Lie superalgebra $\mathcal{G}$. The subclass of finite-dimensional Lie algebras are those which have $S = \emptyset$, and $a_{ii} > 0$ for all $i \in I$. The Cartan matrix is positive-definite, i.e. $\det(A) > 0$ for the finite-dimensional semi-simple Lie algebras. If $a_{ii} > 0$ for all $\imath \in I$, but the Cartan matrix is positive semi-definite, then it is a Kac Moody Lie superalgebra. For a BKM Lie superalgebra the Cartan matrix is not restricted to be positive or positive semi-definite.

The span of each triplet of the form $\{e_i, h_i, f_i\}$, we saw in the case of finite-dimensional semi-simple Lie algebras, was isomorphic to an $sl(2, \mathbb{C})$ algebra. Each element h_i of the Cartan subalgebra, and hence each node in the Dynkin diagram, and each diagonal entry of the Cartan matrix correspond to one such sub-algebra. We will now give a similar decomposition for the case of BKM Lie superalgebras.

Proposition 1 *(i) If* $i \in I\ S$, *and* $a_{ii} \neq 0$, *then the Lie superalgebra* $S_i = \mathbb{C}f_i \oplus \mathbb{C}h_i \oplus \mathbb{C}e_i$ *of the BKM Lie superalgebra* $\mathcal{G}$ *is isomorphic to* $sl(2, \mathbb{C})$.

(ii) If $i \in S$, *then the Lie sub-superalgebra* $S_i = \mathbb{C}[f_i, f_i] \oplus \mathbb{C}f_i \oplus \mathbb{C}h_i \oplus \mathbb{C}[e_i, e_i] \oplus \mathbb{C}e_i$ *is isomorphic to* $sl(0, 1)$.

(iii) If $a_{ii} = 0$, then the Lie sub-(super) algebra $S_i = \mathbb{C}f_i \oplus \mathbb{C}h_i \oplus \mathbb{C}e_i$ is isomorphic to the three-dimensional Heisenberg algebra (resp. superalgebra) if $i \in I$ S(resp $i \in S$

Hence, the BKM Lie superalgebra is generated, like the finite-dimensional Lie algebras we studied in the previous section, by copies of the 3-dimensional Lie algebra $sl(2,\mathbb{C})$, for each even simple root, and of the 5-dimensional Lie superalgebra $sl(0,1)$, for each odd simple root. As before, the adjoint action of each of these $sl(2,\mathbb{C})$ and $sl(0,1)$ on $\mathcal{G}$ decomposes into finite-dimensional represenations. Like before, as a vector space, $\mathcal{G}$ breaks up into the direct sum $\mathcal{G} = N_+ \oplus \mathcal{H} \oplus N_-$, where N_+/N_- are the sub-superalgebras generated by the elements e_i/f_i respectively.

3.5.3 The Root system

The generalized Cartan subalgebra $\mathcal{H}$ acts semi-simply on the BKM Lie superalgebra $\mathcal{G}$ via the adjoint action. This will give us an eigenspace decomposition of $\mathcal{G}$. We used this idea to understand the structure of the finite-dimensional semi-simple Lie algebras. To understand the structure of $\mathcal{G}$ we will look for the eigenvalues and eigenspaces of $\mathcal{H}$. This will give us the root space decomposition of $\mathcal{G}$, and its root system.

Definition 3.5.3 *The formal root lattice Q is defined to be the free abelian group generated by the elements α_i, $i \in I$ with a real valued bilinear form given by $(\alpha_i, \alpha_j) = a_{ij}$. The elements $\alpha_i, i \in I$ are called the simple roots.*

No surprises there. The set of simple roots are defined in a manner very similar to the finite-dimensional Lie algebras. Only, though not apparent, in this case the elements $a_{ij} \in \mathbb{Z}$, unlike in the semi-simple case where they were always equal to 2. There is one more important aspect which makes the root system of a BKM Lie superalgebra very different from that of the other Lie algebras. This is the notion of imaginary simple roots[71]. Let us understand this idea carefully.

For the case of finite-dimensional Lie algebrs, the set of positive roots was finite and all the roots were real (positive definite norm wrt. an inner product defined in the root space). For infinite-dimensional Lie algebras, we saw that there appear a new kind of roots known as *imaginary roots*. However, the simple roots were still all real. For the case of BKM Lie superalgebras Borcherds found that one needs to have imaginary simple roots. This makes the root system of a BKM Lie superalgebra markedly different from the other class of Lie algebras, finite or infinite. We will see how this property alters the denominator identity of BKM Lie superalgebras.

Q, in the general case, may not be an integral lattice, since in general the indexing set I is countably infinite in which case the rank of Q is not finite. Now we come to the root spaces defined by the above root vectors.

Definition 3.5.4 *For $\alpha = \sum_{k=1}^{j} {}_{-}i_k \in Q$, the root space $\mathcal{G}_\alpha$ (resp. $\mathcal{G}_{-\alpha}$) is the subspace of $\mathcal{G}$ generated by the elements $[e_{i_j}[\ldots[e_{i_2}, e_{i_1}]]]$ (resp. $[f_{i_j}[\ldots[f_{i_2}, f_{i_1}]]]$). A non-zero element α of the formal root lattice Q is said to be a root of $\mathcal{G}$ if the subspace G_α is non-trivial. The dimension of the root space $\mathcal{G}_\alpha$ is called the multiplicity of the root α.*

To understand this, compare with the last relation in (3.35). All the root spaces for $i \in I$ are given as $\mathcal{G}_{\alpha_i} = \mathbb{C}e_i$ and $\mathcal{G}_{-\alpha_i} = \mathbb{C}f_i$. In particular, and as before, the root spaces $\mathcal{G}_{\alpha_i}$ and $\mathcal{G}_{-\alpha_i}$ for the simple roots are one-dimensional. Also, as before all the roots $\alpha \in Q$ can be expressed as a sum of simple roots. The root space $\mathcal{G}_\alpha$ is either contained in the even part $\mathcal{G}_0$ or the odd part $\mathcal{G}_1$ of $\mathcal{G}$. There is also the concept of a positive and negative root.

Definition 3.5.5

1. *A root $= a$ is said to be a postive (resp. negative) if α (resp. $-\alpha$) is a sum of simple roots.*
2. *A root α is said to be even (resp. odd) if $\mathcal{G}_\alpha \leq \mathcal{G}_0$ (resp. $\mathcal{G}_1$). We then write $d(\alpha) = 0$ (resp. $d(\alpha) = 1$).*
3. *The height of a root $\alpha = \sum_{k_i} \alpha_i$ is defined to be $\sum k_i$ and is written $ht(\alpha)$.*
4. *The support of α is the set $\{i \in I : k_i \neq 0\}$ and is written $supp(\alpha)$.*
5. *A base of the set of roots L is a linearly independent subset Π such that for any $\alpha \in L, \alpha = \sum_{\beta \in \Pi} k_\beta \beta$, where for all $\beta \in \Pi$, either all the scalars $k_\beta \in \mathbb{Z}_+$ or all $k_\beta \in \mathbb{Z}_-$.*

We recognize the above statements in the context of finite-dimensional semi-simple Lie algebras, but keeping in mind that now we also have imaginary simple roots in the algebra. For any root $\alpha \in L.$, $\text{mult}(\alpha) = \text{mult}(-\alpha)$, and a root α is positive if and only if the root $-\alpha$ is negative. This gives us a decomposition of the set of roots into positive and negative ones. The set of roots L decomposes into $L = L_+ \cup L_-$. As before, this allows us to realize the Cartan decompose on the BKM Lie superalgebra $\mathcal{G}$ as a direct sum of root spaces as

$$\mathcal{G} = (\oplus_{\alpha \in L_+})\mathcal{G}_\alpha \oplus \mathcal{H} \oplus (\oplus_{\alpha \in L-}\mathcal{G}_\alpha) \,. \tag{3.83}$$

Let $\alpha \in L$ and $h_\alpha \in \mathcal{H}$ be such that for all $x \in \mathcal{G}_\alpha$ and $h \in Mh$, $[h, x] = (h_\alpha, h)x$. Then, for all $y \in \mathcal{G}_{-\alpha}$, $[x, y] = (x, y)h_\alpha$.

Before we discuss the WKB denominator formula for BKM Lie superalgebras we will, for the sake of completeness give another characterization of BKM Lie superalgebras given by Borcherds. It is usually very hard to apply the Definition 3.5.2 in terms of the generators and relations to a given Lie algebra to find whether it is a BKM Lie superalgebra or not. Hence it is useful to have different characterizations of BKM Lie superalgebras. The definition below is mainly presented for completeness of our discussion of BKM Lie superalgebras and the need to construct such characterizations may not immediately appear natural. However, systematically following the development of BKM Lie superalgebras will make the reader appreciate the need for such a characterization. It would also be incomplete, however brief a review one constructs, to omit some of the results that helped shape the study of BKM Lie superalgebras. Below we give a characterization of BKM Lie superalgebras.

For a BKM Lie superalgebra the Cartan subalgebra $\mathcal{H}$ is self-centralizing. This property should not appear very surprising from our construction of the Cartan subalgebra for the finite-dimensional semi-simple Lie algebras. An additional property for $\mathcal{H}$ is the existence of a ***regular*** element. This is an element h in $\mathcal{H}$ such that the centralizer[4] of h in $\mathcal{G}$ is $\mathcal{H}$. i.e. $C_{\mathcal{G}}(h) = H$. The existence of a regular element can be used to obtain a bound on the norms of the roots of $\mathcal{G}$. Now, we define a BKM Lie superalgebra in terms of the non-degenerate symmetric bilinear form as follows[72, 73]:

Definition 3.5.6 *Any Lie superalgebra $\mathcal{G}$ satisfying the following conditions is a BKM Lie superalgebra.*

1. *$\mathcal{G}$ has a self centralizing even subalgebra $\mathcal{H}$ with the property that $\mathcal{G}$ is the direct sum of eigenspaces of $\mathcal{H}$, and all the eigenspaces are finite-dimensional.*

2. *There is a non-degenerate invariant supersymmetric bilinear form $(.,.)$ defined on $\mathcal{G}$*

3. *Ther is an element $h \in Mh$ such that $C_{\mathcal{G}}(h) = \mathcal{H}$. If there are only finitely many indices $i \in I$ such that $a_{ii} > 0$, then the norms of the roots of $\mathcal{G}$ are bounded from above. For a given $r \in \mathbb{R}$, there exist only finitely many roots α of $\mathcal{G}$ with $|\alpha(h)| < r$. If $\alpha(h) > 0$ (resp. $\alpha(h) < 0$), α is called a positive (resp. negative) root.*

4. *Let α and β be both positive or both negative roots of non-positive norm. Then $(\alpha, \beta) \leq 0$. Moreover, if $(\alpha, \beta) = 0$ and if $a \in \mathcal{H}_\alpha$ and $[x, \mathcal{G}_\beta] = 0$.*

[4] The centralizer of an element a of a group G, denoted $C_G(a)$ is the set of elements of G which commute with a. $C_G(a) = \{x \in G | \; xa = ax\}$

This completes our discussion on the introduction to BKM Lie superalgebras. We still have one important idea to discuss, though. We will now discuss the Weyl-Kac-Borcherds denominator formula for BKM Lie superalgebras. Considering the importance of this idea to this thesis, we have saved the discussion on the denominator formula till after we have all the ideas required to construct it. The denominator identity occurs in the representation theory of Lie algebras as the special case of the Weyl character formula. We start with a discussion on the character theory of Lie algebras to motivate the character and denominator formulas.

3.6 Denominator Identities

3.6.1 Characters Of Irreducible Representations

We start with the finite-dimensional semi-simple Lie algebras and then graduate to the infinite dimensional ones to give the reader a better understanding of the various aspects of the denominator identity and how they get modified as one considers the more non-trivial class of Lie algebras. We go back to the representations of $sl(2, \mathbb{C})$ and $sl(3, \mathbb{C})$ in that we studied earler to study the concept characters of representations. One of the offshoots of this is the denominator identity, which will be very crucial to our problem of counting BPS states in string theory. The motivation for character theory is as follows. Given two representations, V and V' (that is, $\rho : \mathfrak{g} \to gl(V)$ and $\rho' : \mathfrak{g} \to gl(V')$) of a Lie algebra $\mathfrak{g}$, we say that V is *isomorphic* or *equivalent* to V', if there is an isomorphism of vector spaces $T : V \to V'$ which is compatible with the operation of $\mathfrak{g}$:

$$\rho'_g T(v) = T(\rho_g(v)), \tag{3.84}$$

for all $v \in V$ and all $g \in \mathfrak{g}$.

A given representation usually has a complicated description in terms of matrices, and it is not always apparently obvious if two given representations are isomorphic to each other or not. It would be useful to have a way of determining such relationships between representations without having to go into the details of the representations. Suppose we could construct a quantity, say a function, that captures some intrinsic quality of the representation and is sufficient to determine whether or not two representations are isomorphic to each other just by comparing the value of the function on the given representation. Speaking mathematically, we need a class function (A function that is invariant over a conjugacy class, which in our case, are the isomorphism classes of irreducible representations) that characterizes isomorphic representations. This leads to the idea of the ***character*** of an irreducible representation of a semi-simple Lie algebra $\mathfrak{g}$ which we study now.

Definition 3.6.1 *The character χ_μ of a finite-dimensional irreducible representation, with highest weight μ, is defined as the map from $\mathfrak{h} \to \mathbb{C}$ given by*

$$\chi_\mu(x) : h \mapsto \chi_\mu(h) = Tr\ \exp(\rho(x)) \ . \tag{3.85}$$

The character of a finite-dimensional representation determines the represenation up to equivalence. The function depends only on the equivalence class of ρ and satisfies

$$\chi_\mu(gxg^{-1}) = \chi_\mu \tag{3.86}$$

Here, we have defined the character to be a map from $\mathfrak{h}$ to $\mathbb{C}$. We could also consider weights, μ, in place of $h \in \mathfrak{h}$ as arguments of χ_μ. From eq (3.39) we see that $\rho(h) \cdot v_\mu = \langle \mu, h \rangle v_\mu$ for all weight vectors $v_\mu \in V$ and we can rewrite the character χ_μ for a weight μ as

$$\chi_\mu(h) = \sum_\mu \ \mathrm{mult}(\mu) \ \exp(\langle \mu, h \rangle), \tag{3.87}$$

where the sum is over the set of all weights in V.

The operator $\rho(0)$ is a $d_V \times d_V$ matrix with all entries equal to zero, where d_V is the dimension of the representation V. The character $\chi_\mu(0)$ evaluated on the zero weight gives the dimension on the representation

$$\chi_\mu(0) = d_V \ .$$

The character for the direct sum of two representations is equal to the sum of the characters of the constituent representations. Similarly, the character of a quotient of representations is obtained by subtracting the character of the submodule which is quotiented out from the character of the original representation.

We can use the action of the Weyl group on the set of weights to express the character of a highest weight module (representation) of $\mathfrak{g}$. It is called the ***Weyl character formula***.

Let V be an irreducible finite-dimensional representation of the complex semi-simple Lie algebra $\mathfrak{g}$ with highest weight μ. Then

$$\chi_\mu(h) = \frac{\sum_{w\in\mathcal{W}}(-1)^{l(w)}\exp[\langle w(\mu+\rho), h\rangle]}{\sum_{w\in\mathcal{W}}(-1)^{l(w)}\exp[\langle w(\rho), h\rangle]}, \tag{3.88}$$

where ρ is the Weyl vector as defined in (3.29) or (3.30), and the sum is over the full Weyl group. Thus, one can compute the character of an irreducible finite-dimensional representation from the knowledge of the action of the Weyl group on the elements. Now we discuss the denominator identity.

3.6.2 The Denominator Identity

Consider the denominator of eq. (3.88)

$$\begin{aligned}\sum_{w\in\mathcal{W}}(-1)^{l(w)}\exp[\langle w(\rho), h\rangle] &= \prod_{\alpha\in L_+}\left[\exp(\tfrac{1}{2}\langle\alpha, h\rangle) - \exp(-\tfrac{1}{2}\langle\alpha, h\rangle)\right] \\ &= \exp(\langle\rho, h\rangle)\prod_{\alpha\in L_+}\left[1-\exp(-\langle\rho, h\rangle)\right].\end{aligned} \tag{3.89}$$

This is known as the ***denominator identity***. The Weyl denominator formula is a specialisation of the Weyl character formula to the trivial representation. Conventionally, the denominator formula is written as (3.89), but for our purpose of generalizing it to infinite-dimensional Lie algebras, we recast it into a form better suited for the generalization.

$$\prod_{\alpha\in L_+}\left(1-\exp\left(-\langle\alpha, h\rangle\right)\right) = \sum_{w\in\mathcal{W}}\det(w)\,\exp\left(w(\langle\rho, h\rangle) - \langle\rho, h\rangle\right)\ , \tag{3.90}$$

where $w(\rho)$ is the image of ρ under the action of the element w of the Weyl group. The importance of the above formula, from both the general and the point of view of our problem, cannot be overstated. It is at the heart of the relation between the spectrum of $\frac{1}{4}$-BPS states and BKM Lie superalgebras. Let us look at eq. (3.89) more closely and see what it contains that makes it so important. Given the RHS of eq. (3.89), we have knowledge of all the positive roots of $\mathfrak{g}$ and their respective multiplicities and given the LHS we have knowledge of the Weyl group of $\mathfrak{g}$ and its action on all the roots. Thus, the denominator identity contains all the essential information about the Lie algebra $\mathfrak{g}$ and given the denominator identity, one can construct $\mathfrak{g}$ completely from it. In the theory of BKM Lie superalgebras it plays a central role not only because it contains the information of $\mathcal{G}$ in it, but also because it provides the link with automorphic forms.

Let us start by computing the denominator identity of $sl(3,\mathbb{C})$. As discussed before, $sl(3,\mathbb{C})$ has two simple roots α_1 and α_2. The set of positive roots, L_+, is given by α_1, α_2 and $\alpha_3 = \alpha_1 + \alpha_2$. The multiplicity of each positive root is one. The Weyl group is given by the permutation group of three elements, S_3. The elements are the reflections r_1 and r_2 , with respect to the two simple roots. The action of the reflections on the roots is given by

$$\begin{aligned} w_{\alpha_i}(\alpha_i) &= -\alpha_i \ , \\ w_{\alpha_i}(\alpha_j) &= (\alpha_i + \alpha_j) \end{aligned} \tag{3.91}$$

The elements of S_3, generated by w_{α_1} and w_{α_2} are given by

$$(1, w_{\alpha_1}, w_{\alpha_2}, w_{\alpha_1} \cdot w_{\alpha_2}, w_{\alpha_2} \cdot w_{\alpha_1}, w_{\alpha_1} \cdot w_{\alpha_2} \cdot w_{\alpha_1}) \tag{3.92}$$

The action of the six elements on ρ is $(\rho, -\alpha_1, -\alpha_2, -\rho, \alpha_1, \alpha_2)$. Putting it all together into eq (3.89) the Weyl denominator formula for $sl(3,\mathbb{C})$ is given by

$$\prod_{\phi \in L_+} (1 - e(-\phi)) = \sum_{w \in W} (-1)^{\mathrm{l}(w)} e(w(\rho) - \rho). \tag{3.93}$$

where ρ is the Weyl vector, and w is an element of the Weyl group W. Denoting $u = e(-\alpha_1)$ and $v = e(-\alpha_2)$, we get

$$\text{LHS} = (1-u)(1-v)(1-uv) = \left[1 - u - v + u^2 \cdot v + u \cdot v^2 - u^2 \cdot v^2\right] \tag{3.94}$$

Now consider the RHS.

$$\text{RHS} = \left[1 + u^2 \cdot v + u \cdot v^2 - u^2 \cdot v^2 - u - v\right] \tag{3.95}$$

where $\mathrm{l}(w) = +1$ for $w = 1, x, x^2$ and -1 otherwise. The equality is obvious.

This completes our discussion of the denominator formula for finite-dimensional semi-simple Lie algebras. We now look at the denominator identity of affine Lie algebras. We discussed the Weyl group of affine Lie algebras when in Section 3.4.3. As in the finite-dimensional case, one of the main applications of the Weyl group is the calculation of characters of highest weight modules, and the denominator formula which is an offshoot of the

Weyl character formula. As before, the characters χ_μ are defined as

$$\chi_\mu = \sum_\lambda \text{mult}(\mu)\ e^\lambda, \tag{3.96}$$

where we have defined the $\exp\langle\lambda, h\rangle = e^\lambda$ as formal exponenials. The ***Weyl-Kac character formula*** [3]

$$\chi_\mu(h) = \frac{\sum_{w\in\widehat{\mathcal{W}}}(-1)^{l(w)}\mathrm{e}^{w(\mu+\rho)}}{\sum_{w\in\widehat{\mathcal{W}}}(-1)^{l(w)}\mathrm{e}^{w(\rho)}}, \tag{3.97}$$

where ρ is the Weyl vector defined as before.

The denominator identity for the case of affine Lie algebras becomes

$$\sum_{w\in\widehat{\mathcal{W}}}(-1)^{l(w)}\mathrm{e}^{w(\rho)} = \mathrm{e}^\rho \prod_{\widehat{\alpha}\in\widehat{L}_+}(1-\mathrm{e}^{-\widehat{\alpha}})^{\text{mult}(\widehat{\alpha})}\ . \tag{3.98}$$

The multiplicities of all the roots in the finite-dimensional case were 1, and the term on the right hand side, therefore, did not have the multiplicity factor.

An alternate definition is given by Lepowsky and Milne which is tailored to writing the sum side of the denominator formula. The key observation (due to MacDonald) is that $[w(\rho)-\rho]$ behaves better than either of the terms. Recall that an element of the Weyl group acts as a permutation of all roots (not necessarily positive). Thus, $[w(\rho)-\rho]$ obtains contribution, only when a positive root gets mapped to a non-positive root. So one defines the set Φ_w for all $w\in\widehat{\mathcal{W}}$,

$$\widehat{\Phi}_w = w(\widehat{L}_-)\cap\widehat{L}_+ = \left\{\hat{\alpha}\in\widehat{L}_+\ \middle|\ w^{-1}(\hat{\alpha})\in\widehat{L}_-\right\}\ . \tag{3.99}$$

Using this definition, we can see that

$$\rho - w(\rho) = \frac{1}{2}\sum_{\hat{\alpha}\in\widehat{L}_+}[\hat{\alpha}-w(\hat{\alpha})] \sim \langle\widehat{\Phi}_w\rangle\ , \tag{3.100}$$

where $\langle\widehat{\Phi}_w\rangle$ is the sum of elements of the set $\widehat{\Phi}_w$. Note that $-\widehat{L}_- = \widehat{L}_+$, which explains the half disappearing in the RHS of the above formula. Imaginary roots do not appear in the set $\widehat{\Phi}_w$ for affine Lie algebras as the imaginary roots turn out to be Weyl invariant and hence cancel out in the above equation.

The denominator formula that works for affine Kac-Moody algebras, after including the imaginary roots in $\widehat{L}_+$, is the Weyl-Kac denominator formula

$$\prod_{\hat{\alpha}\in L_+} \left(1-e^{-\hat{\alpha}}\right)^{\text{mult}(\hat{\alpha})} = \sum_{w\in\hat{\mathcal{W}}} \det(w)\ e^{-\langle\hat{\Phi}_w\rangle}\ , \tag{3.101}$$

We now consider the example of an affine Lie algebra, $\widehat{sl(n,\mathbb{C})}$, given by the root system ($A_l^{(1)}$, $\quad l=n-1$). We derive the general expressions for $A_l^{(1)}$, then specialize them to the case of$A_1^{(1)}$ and $A_2^{(1)}$ for the sake of illustration. The set of simple roots of $A_l^{(1)}$ are given by the simple roots of the horizontal subalgebra, A_l, together with the root $\delta-\mu$, where δ is the smallest imaginary root, and μ is the highest root of A_l (see eq (3.60).

The set of roots is given by functionals of the form $j\delta+\mu$, where $j\in\mathbb{Z}$ and $\mu\in\widehat{L}$. The imaginary roots are given by functionals of the form $j\delta$, where $j\in\mathbb{Z}, j\neq 0$. We define the set of positive roots as the union of the set of positive roots of the horizontal subalgebra $\widehat{L}$, with the set of roots in $\widehat{L}$ which have positive eigenvalues w.r.t d. Thus the set of positive roots of $A_l^{(1)}$ are given by,

$$\widehat{L}_+ = \{(s-1)\cdot\delta+\hat{\alpha}_i+...\hat{\alpha}_{i+k-1}, s\cdot\delta-(\hat{\alpha}_i+...\hat{\alpha}_{i+k-1}), s\cdot\delta \mid 1\leq k\leq l; s\in\mathbb{Z}_+-\{0\}\}. \tag{3.102}$$

The real roots have multiplicity 1 and the imaginary roots have multiplicity l. The denominator formula is given by:

$$\prod_{\hat{\alpha}\in\hat{L}_+} (1-e(-\hat{\alpha}))^{\text{mult}(\hat{\alpha})} = \sum_{w\in\widehat{W}} (-1)^{l(w)} e(w(\rho)-\rho)\ . \tag{3.103}$$

Recasting it as eq. (3.101) we have

$$\prod_{\hat{\alpha}\in\hat{L}_+} (1-e(-\hat{\alpha}))^{\text{mult}(\hat{\alpha})} = \sum_{w\in\widehat{W}} (-1)^{l(w)} e(-\langle\widehat{\Phi}_w\rangle) \tag{3.104}$$

where $\widehat{\Phi}_w = \hat{L}_+\cap\hat{L}_-$, and $\langle\widehat{\Phi}\rangle$ is the sum of all elements of $\widehat{\Phi}$. We know the simple and positive roots of the Lie algbra. We now need to evaluate the sets $\widehat{\Phi}_w$ and the Weyl group to compute the denominator formula. To determine the set $\widehat{\Phi}$, we recall the action of the elements of the Weyl group on the set of roots.

$$\widehat{W}(\widehat{L}_R) = \widehat{L}_R \quad \widehat{W}(\widehat{L}_I) = \widehat{L}_I, \tag{3.105}$$

and

$$\widehat{W}(\widehat{L}_I \cap \widehat{l}_+) = (\widehat{L}_I \cap \widehat{L}_+) \tag{3.106}$$

. The set $\widehat{\Phi}_w$ is the set of all roots $\{\widehat{\alpha} \in \widehat{L}_+ \mid w^{-1}\alpha \in \widehat{L}_-\}$. Thus, $\widehat{\Phi}_w$ consists of elements of the form

$$\{\beta, \beta + n.\delta\}, \text{where}, \beta \in \widehat{L}_+ \text{ and } m \in \mathbb{Z}_+ \tag{3.107}$$

or

$$\{\beta + n.\delta\}\text{where}, \beta \in \widehat{L}_- \text{ and } n \in \mathbb{Z}_+ - \{0\}\ . \tag{3.108}$$

3.6.3 Denominator Identity for $\widehat{sl(2, \mathbb{C})}$

For the affine Kac-Moody algebra, $\widehat{A}_1^{(1)}$, from the above definition of the set of positive roots, we have

$$\widehat{L}_+ = \Big(n(\widehat{\alpha}_1 + \widehat{\alpha}_0), n\widehat{\alpha}_1 + (n-1)\widehat{\alpha}_0, (n-1)\widehat{\alpha}_1 + n\widehat{\alpha}_0 \ \Big|\ n = 1, 2, 3, \ldots \Big)\ , \tag{3.109}$$

and the Weyl group is isomorphic to $\mathbb{Z}_2 \ltimes \mathbb{Z}$. Putting it all together into the denominator identity gives

$$\prod_{n\geq 1}(1 - e^{-n\alpha_0}e^{-n\alpha_1})(1 - e^{-(n-1)\alpha_0}e^{-n\alpha_1})(1 - e^{-n\alpha_0}e^{-(n-1)\alpha_1})$$
$$= \sum_{n\in\mathbb{Z}} e^{-n(2n-1)\alpha_0}e^{-n(2n+1)\alpha_1} - \sum_{n\in\mathbb{Z}} e^{-(n+1)(2n+1)\alpha_0}e^{-n(2n+1)\alpha_1} \tag{3.110}$$

Setting $e^{-\alpha_0} = r$ and $e^{-\alpha_1} = qr^{-1}$, the above identity is equivalent to the Jacobi triple identity involving the theta function $\vartheta_1(\tau, z)$:

$$\begin{aligned} -i\vartheta_1(\tau, z) &= q^{1/8}r^{-1/2}\prod_{n=1}^{\infty}(1-q^n)\ \big(1-q^{n-1}r\big)\ \big(1-q^n r^{-1}\big) \\ &= \sum_{n\in\mathbb{Z}}(-1)^n\ q^{\frac{(n-1/2)^2}{2}}\ r^{n-1/2}\ . \end{aligned} \tag{3.111}$$

3.6.4 Denominator Formula for $\widehat{sl(3, \mathbb{C})}$.

We apply the above ideas to the case of $\widehat{sl(3, \mathbb{C})}$ ($A_2^{(1)}$) as an example and write down the denominator identity for it[74, 75, 76, 77] .

The horizontal algebra for $\widehat{sl(3,\mathbb{C})}$ is $sl(3,\mathbb{C})$. There are two elements in the Cartan subalgebra, $\widehat{\alpha}_1$, and $\widehat{\alpha}_2$. The Weyl group of $sl(3,\mathbb{C})$ is S_3, generated by two elements. It is generated by the reflections with respect to the two simple roots $\widehat{\alpha}_1$, and $\widehat{\alpha}_2$ (call the reflections $w_{\widehat{\alpha}_1}$ and $w_{\widehat{\alpha}_2}$ respectively). The simple roots of the affine Lie algebra $\widehat{sl(3,\mathbb{C})}$ are given by the simple roots of the horizontal algebra, together with $\widehat{\alpha}_0 = \delta - (\widehat{\alpha}_1 + \widehat{\alpha}_2)$, where δ is the smallest positive imaginary root of $A_2^{(1)}$.

The Weyl group of $\widehat{sl(3,\mathbb{C})}$ is the semi direct product of the Weyl group of $sl(3,\mathbb{C})$ and an abelian group, T, of translations generated by two elements ($\cong \mathbb{Z}^2$).

Let $t(m,n) \in T$ be an allowed translation whose action on $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ is given by:

$$t\widehat{\alpha}_1 = \widehat{\alpha}_1 + m\delta$$

$$t\widehat{\alpha}_2 = \widehat{\alpha}_2 + n\delta \tag{3.112}$$

It follows that

$$t\widehat{\alpha}_0 = \widehat{\alpha}_0 + q\delta, \tag{3.113}$$

such that $(m+n+q) = 0$. The elements of the Weyl group are of the form $w_{\widehat{\alpha}} = w_\alpha.t$ where $w_\alpha \in S_3$ and $t(m,n) \in T$. Let $\widehat{W}_t$ be the subgroup of $\widehat{W}$ genrated by $t(m,n)$ (written as t for brevity hence forth). Thus,

$$\widehat{W} = \widehat{W}_L \cup w_{\widehat{\alpha}_1} \cdot \widehat{W}_L \cup w_{\widehat{\alpha}_2} \cdot \widehat{W}_L \cup w_{\widehat{\alpha}_1} \cdot w_{\widehat{\alpha}_2} \cdot \widehat{W}_L \cup w_{\widehat{\alpha}_2} \cdot w_{\widehat{\alpha}_1} \cdot \widehat{W}_L \cup \hat{w}_{\hat{\alpha_1}} \cdot \hat{w}_{\hat{\alpha_2}} \cdot \hat{w}_{\hat{\alpha_1}} \cot \widehat{W}_L, \tag{3.114}$$

for $w_{\widehat{\alpha}_1}, w_{\widehat{\alpha}_2} \in S_3$ and $t \in T$.

Now, to compute the denominator formula, we need to determine the action of the Weyl group on $\langle \widehat{\Phi}_t \rangle$. From the definition of $\widehat{\Phi}$, and the action of the elements of the Weyl group on the set of roots, we have,

$$\widehat{\Phi}_t = \{\widehat{\alpha}_1 + i\delta, \widehat{\alpha}_2 + j\delta, \widehat{\alpha}_1 + \widehat{\alpha}_2 + k\delta \mid 0 \le i \le (m-1), 0 \le j \le (n-1), 0 \le k \le (q-1)\}\,. \tag{3.115}$$

Thus,

$$\langle \widehat{\Phi}_t \rangle = (m+k)\widehat{\alpha}_1 + (n+k)\widehat{\alpha}_2 + \frac{[m(m-1) + n(n-1) + k(k-1)]}{2}\delta \tag{3.116}$$

Putting all the above together in the denominator formula, we have:

$$\begin{aligned}
\prod_{s\geq 1}&(1-u_0^s.u_1^s.u_2^s)^2(1-u_0^{s-1}.u_1^s.u_2^{s-1})(1-u_0^{s-1}.u_1^{s-1}.u_2^s)\\
&\times(1-u_0^{s-1}.u_1^s.u_2^s)(1-u_0^s.u_1^{s-1}.u_2^s)(1-u_0^s.u_1^s.u_2^{s-1})\\
=&\sum_{r_j\equiv 0 (\mathrm{mod}\ 3)} u_0^{\frac{1}{6}(r_1(r_1-2)+r_2^2+r_3(r_3+2))}u_1^{\frac{1}{6}(r_1^2+r_2(r_2+2)+r_3(r_3-2))}u_2^{\frac{1}{6}(r_1(r_1+2)+r_2(r_2-2)+r_3^2)}\\
-&\sum_{r_1=1,r_2=0,r_3=2 (\mathrm{mod}\ 3)} u_0^{\frac{1}{6}(r_1(r_1+2)+r_2^2+r_3(r_3-2))}u_1^{\frac{1}{6}(r_1(r_1-2)+r_2(r_2+2)+r_3^2)}u_2^{\frac{1}{6}(r_1^2+r_2(r_2-2)+r_3(r_3+2))}\\
+&\sum_{r_j\equiv 1 (\mathrm{mod}\ 3)} u_0^{\frac{1}{6}(r_1(r_1+2)+r_2(r_2-2)+r_3^2)}u_1^{\frac{1}{6}(r_1(r_1-2)+r_2^2+r_3(r_3+2))}u_2^{\frac{1}{6}(r_1^2+r_2(r_2+2)+r_3(r_3-2))}\\
-&\sum_{r_1=0,r_2=2,r_3=1 (\mathrm{mod}\ 3)} u_0^{\frac{1}{6}(r_1(r_1-2)+r_2(r_2+2)+r_3^2)}u_1^{\frac{1}{6}(r_1^2+r_2(r_2-2)+r_3(r_3+2))}u_2^{\frac{1}{6}(r_1(r_1+2)+r_2^2+r_3(r_3-2))}\\
+&\sum_{r_j\equiv 2 (\mathrm{mod}\ 3)} u_0^{\frac{1}{6}(r_1^2+r_2(r_2+2)+r_3(r_3-2))}u_1^{\frac{1}{6}(r_1(r_1+2)+r_2(r_2-2)+r_3^2)}u_2^{\frac{1}{6}(r_1(r_1-2)+r_2^2+r_3(r_3+2))}\\
-&\sum_{r_1=2,r_2=1,r_3=0 (\mathrm{mod}\ 3)} u_0^{\frac{1}{6}(r_1^2+r_2(r_2-2)+r_3(r_3+2))}u_1^{\frac{1}{6}(r_1(r_1+2)+r_2^2+r_3(r_3-2))}u_2^{\frac{1}{6}(r_1(r_1-2)+r_2(r_2+2)+r_3^2)}\ ,
\end{aligned}$$

where $r_1, r_2, r_3 \in \mathbb{Z}$, and $m = \frac{1}{3}(r_2 - r_3)$; $n = \frac{1}{3}(r_1 - r_2)$; $q = \frac{1}{3}(r_3 - r_1)$ That completes our study of the denominator identity for affine Lie algebras. We see how the modifications that occur due to the presence of the imaginary root. We next study the denominator identity for BKM Lie superalgebras. This was first constructed by Borcherds. We will state the denominator identity and the super-denominator identity for BKM Lie superalgebras and explain how it is obtained as a generalization of the denominator identity for affine Lie algebras. Discussing examples of BKM Lie superalgebras is beyond the scope of this work and the reader is refered to the literature.

To define the Weyl-Kac-Borcherds denominator formula we first need to define the (even) Weyl group of a BKM Lie superalgebra. As before, we define the reflection w_α along a hyperplane perpendicular to α when α is an even (resp. odd) root of non-zero norm. For all weights $\beta \in \mathcal{H}$ r $\mathbb{C} \otimes_{\mathbb{Z}} Q$

$$w_\alpha(\beta) = \begin{cases} \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} & \text{if } \deg(\alpha) = 0\ , \\ \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} & \text{if } \deg(\alpha) = 1\ . \end{cases} \tag{3.117}$$

Here for the above formula to hold the roots α are also required to satisfy athe additional condition that for any $x \in \mathcal{G}_\alpha, y \in \mathcal{G}$, there is a non-negative integer n depending on x and y such that $(\mathrm{ad}x)^n y = 0$. The Weyl group $\mathcal{W}_E$ is defined as follows

Definition 3.6.2 *The even Weyl group $\mathcal{W}_E$ is defined to be the group generated by the reflections $w_{\alpha_i}, i \in I$ such that $a_{ii} > 0$[3]. The Weyl group $\mathcal{W}$ is generated by all the reflections w_{α_i} where $\alpha_i \in L^+$ and satisfies* (3.117) *and is of non-zero norm.*

For all infinite dimensional BKM Lie superalgebras the groups $\mathcal{W}_E$ and $\mathcal{W}$ are the same. The Weyl vector for BKM Lie superalgebras is defined as follows:

Definition 3.6.3 *A Weyl vector is defined to be a vector ρ either in the dual space $\mathbb{C} \otimes_{\mathbb{Z}} Q$ or in $\mathcal{H}$ or in its dual $\mathcal{H}^*$ satisfying*

$$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) \text{ for all } i \in I \tag{3.118}$$

We first need to find and expression for the dimensions of the weight spaces of the super algebra to be able to differentiate the odd and even weight spaces. We thus, require to find the character and super-character for the super-algebra. For this, we need to work with formal exponentials e^λ. We define the character and supercharacter using formal exponentials as follows:

Definition 3.6.4 *Let ε be the commutative associative algebra of formal series*

$$\sum_{\lambda \in \mathcal{H}} x_\lambda e^\lambda$$

for which there exist finitely many elements $\lambda_i \in \mathcal{H}$, $i = 1, \ldots, m$ such that the coefficients x_λ are non-trivial only if $\lambda \leq \lambda_i$ for some $1 \leq i \leq m$. Multiplication is defined by $e^\lambda e^\mu = e^{\lambda+\mu}$. The character and super-character of the $\mathcal{H}$-module $V = V_0 \oplus V_1 \in \mathcal{O}$ are the elements of the algebra ε defined respectively to be the formal sums:

$$ch\ V = \sum_{\lambda \in \mathcal{H}} dim\ V_\lambda e^\lambda) \quad \text{and} \quad sch\ V = \sum_{\lambda \in \mathcal{H}} (dim\ V_{0_\lambda} - dim\ V_{1_\lambda}) e^\lambda\ . \tag{3.119}$$

where $V(\Lambda) \in \mathcal{O}$ is a highest weight module of highest weight Λ, it is assumed that $deg(\Lambda) = 0$.

Now we write the denominator identities that the above character and super-character formulae lead to. For $\mu = \sum_{i \in I} k_i \alpha_i$, let us call $\sum_{i \in I \setminus S} k_i$ as $\mathrm{ht}_0(\mu)$, and $\sum_{i \in I} k_i$ as $\mathrm{ht}(\mu)$.

We define the following

$$T_\Lambda = e(\Lambda + \rho) \sum \epsilon(\mu) e^{-\mu} \quad \text{and} \quad T'_\Lambda = e^{\Lambda+\rho} \sum \epsilon'(\mu) e^{-\mu}, \tag{3.120}$$

where

$$\epsilon(\mu) = (-1)^{\mathrm{ht}(\mu)} \quad \text{and} \quad \epsilon'(\mu) = (-1)^{\mathrm{ht}_0(\mu)} . \tag{3.121}$$

In terms of the above definitions, we define the denominator and super-denominator formula for any BKM Lie superalgebra.

Definition 3.6.5 *For any BKM Lie superalgebra $\mathcal{G}$,*

$$\frac{\prod_{\alpha \in L_0^+} (1 - e^{-\alpha})^{mult_0(\alpha)}}{\prod_{\alpha \in L_1^+} (1 + e^{-\alpha})^{mult_1(\alpha)}} = e^{-\rho} \sum_{w \in \mathcal{W}} \det(w)\ w(T), \tag{3.122}$$

and

$$\frac{\prod_{\alpha \in L_0^+} (1 - e^{-\alpha})^{mult_0(\alpha)}}{\prod_{\alpha \in L_1^+} (1 - e^{-\alpha})^{mult_1(\alpha)}} = e^{-\rho} \sum_{w \in \mathcal{W}} \det(w)\ w(T') \tag{3.123}$$

are respectively the denominator formula and the super-denominator formula.

That completes our definition of the denominator and super-denominator formulae for the case of BKM Lie superalgebras.

3.6.5 The Fake Monster Lie Algebra

As an example of the above formula in the setting of a BKM Lie superalgebra, we will briefly discuss the example of the fake monster Lie algabra [61, 57] which is a BKM Lie algebra describing the physical states of a bosonic string on a torus. Its root lattice is a 26 dimensional even unimodular Lorentzian lattice[5] denoted $II_{25,1} = \Lambda \oplus II_{1,1}$ where Λ is the Leech lattice with elements $\alpha = (\lambda, m, n)$ ($\lambda \in \Lambda$ and $(m,n) \in II_{1,1}$) with norm $\alpha^2 = \lambda^2 - 2mn$ (it is the unique positive definite lattice of rank 24 with no norm 2 vectors [78]). and $II_{1,1}$ is the unique even unimodular Lorentzian lattice of rank 2.

The roots of $II_{25,1}$ are the non-zero vectors α with $\alpha^2 \leq 2$. Their multiplicity is given by $p_{24}(1 - \alpha^2/2)$, where $p_{24}(n)$ is the number of partitions of n into parts of 24 colors. Thus,

[5] An integral lattice L is said to be *even* if for all $v \in L$, $(v,v) \equiv 0$ (mod2). Else it is said to be *odd*. The *dimension* and *signature* of L are the dimension and signature, respectively, of the real vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$ with the bilinear form induced from L. A lattice is called *Lorentzian* if it has signature $(m,1)$ or $(1,m)$. A lattice is a unimodular one if $L = L^*$, where L^* is the dual of L.

the multiplicities of the roots are given by

$$\sum_n p_{24}(1+n)q^n = 1/\Delta(q) = q^{-1}\prod_{n>0}(1-q^n)^{-24} = q^{-1} + 24 + 324q + 3200q + \dots \ . \quad (3.124)$$

The real simple roots are the norm 2 vectors α in $II_{25,1}$, which are in bijective correspondence with points

$$(\lambda, 1, \tfrac{\lambda^2}{2} - 1), \quad \lambda \in \Lambda \quad (3.125)$$

in the Leech lattice. They all satisfy $(\rho, \alpha) = 1$ for the Weyl vector $\rho = (0, 0, -1)$. The imaginary simple roots are of the form

$$(0, 0, n), \quad n \in \mathbb{N} \quad (3.126)$$

and they all have multiplicity $p_{24}(1) = 24$. These satisfy $(\rho, \alpha) = 0$ for their inner product with the Weyl vector.

The set of positive roots are given by the set of roots $\alpha = (\lambda, m, n)$ such that $m > 0$ or $\alpha = (0, 0, n)$. Thus, the positive roots are

$$\alpha \in L^+ = \{\alpha \in II_{25,1} | (\alpha, \rho) > 0 \ or \ \alpha = (0, 0, n)\} \quad (3.127)$$

The Weyl group of the algebra is generated by the real simple roots with norm 2, and thus the Weyl group of $II_{25,1}$ is isomorphic to the reflection group of the Leech lattice.

Now we can write down the denominator identity of the fake monster Lie algebra from the above information as follows. Given the set of positive roots (3.127) and the fact that they all have multiplicity $p_{24}(1 - \alpha^2/2)$, we can write down the product side of the denominator identity as

$$\prod_{\alpha \in L^+} (1 - e^{-\alpha})^{p_{24}(1-\alpha^2/2)} \ . \quad (3.128)$$

The Weyl group is known and hence we can form the sum side of the denominator identity uning the fact that all the imaginary simple roots have norm 0 and are mutually orthogonal. The sum side is given by

$$\begin{aligned} &\sum_{n_1, n_2, \dots} (-1)^{n_1+n_2+\dots} e^{n_1\rho} \binom{24}{n_1} e^{n_2\rho} \binom{24}{n_2} \dots \\ = \ &(1 - e^{\rho})^{24}(1 - e^{2\rho})^{24} \dots \ . \end{aligned} \quad (3.129)$$

Putting the above two equations together gives

$$e^{\rho}\prod_{\alpha\in L^+}(1-e^{-\alpha})^{p_{24}(1-\alpha^2/2)}=\sum_{\substack{w\in\mathcal{W}\\ n\in\mathbb{Z}}}\det(w)w\Big(e^{\rho}\prod_{n>0}(1-e^{n\rho})^{24}\Big)\ . \tag{3.130}$$

For further examples the reader is referred to the mathematical literature[71, 79, 80]. We will see other examples of BKM Lie superalgebras in Chapter 6 when we construct the BKM Lie superalgebras corresponding to the modular forms occuring in the CHL strings.

3.7 Conclusion

In this chapter we have studied the theory of Lie algebras covering finite-dimensional semi-simple Lie algebras, affine Lie algebras, and BKM Lie superalgebras. We have seen how starting from the finite-dimensional Lie algebras the various constructions are modified and generalized to finally get BKM Lie superalgebras. The presence of imaginary roots differentiates the infinite-dimensional Lie algebras from the finite-dimensional ones, while the presence of imaginary simple roots is a characterstic of the BKM Lie superalgebras. We will put these ideas to use later in the problem of counting black hole microstates.

4

Modular Forms

4.1 Preliminary Definitions:

1. **Holomorphic Function:** The concept of a holomorphic function (also known as an analytic function) extends the concept of real functions of real variables to complex functions of complex variables. Let z_0 be a point in $\mathbb{C}$ and f a function on $\mathbb{C}$. We say that f is complex-differentiable at the point z_0, if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.1}$$

exists. For a complex valued function, this is equivalent to the *Cauchy-Riemann* conditions on the real and imaginary parts of the complex function.

Now, let $\mathbb{U}$ be an open subset of $\mathbb{C}$. A function $f : \mathbb{U} \to \mathbb{C}$ is said to be ***holomorphic*** if f takes values in $\mathbb{C}$, and is complex-differentiable at every point in $\mathbb{U}$.

The sum and product of two holomorphic functions is again a holomorphic function. The same is true of the quotient of two holomorphic functions whenever the denominator is non-vanishing. The derivative of a holomorphic function is itself holomorphic. Thus, holomorphic functions are infinitely differentiable and can be described by their Taylor series.

Some readers may be be familiar with the definition of holomorphic functions as functions that depend on the variable z alone, and are independent of $\bar{z}$. The functions defined above, when written in terms of z and $\bar{z}$, can be seen to be dependent only on the variable z and thus represent the same thing.

2. **Meromorphic Function:** The quotient of two holomorphic functions, we said, is again a holomorphic function. Such a function will be holomorphic whenever the denominator is non vanishing. This leads to the notion of a ***meromorphic*** function. A function f, on an open subset $\mathbb{U}$ of the complex plane, is said to be ***meromorphic*** if it is holomorphic on $\mathbb{U}$ execpt at a discrete set of points in $\mathbb{U}$ which are the ***poles*** of the function. The poles are just the set of points where the denominator vanishes. The poles of a meromorphic function are isolated. The sum, product and the ratio of two meromorphic functions is again a meromorphic function.

3. $SL(N, \mathbb{F})$ **(Special Linear Group):** It is the set of all $N \times N$ matrices with entries in the field $\mathbb{F}$, and determinant 1. It is a simple Lie group. It is a subgroup of the general linear group over the field $\mathbb{F}$, which is the group of all $n \times n$ invertible matrices, with entries from $\mathbb{F}$, together with the operation of matrix multiplication. We will mostly study the Lie group $SL(2, \mathbb{Z})$, which is over the field of integers, and some of its discrete subgroups

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F} \text{ and } ad - bc = 1 \right\} . \tag{4.2}$$

4. **Upper Half Plane:** The upper half plane, $\mathfrak{H}$, is the set of complex numbers with positive imaginary parts. i.e.

$$\mathfrak{H} = \{x + iy \mid y > 0; x, y \in \mathbb{R}\} . \tag{4.3}$$

   It is a Riemannian manifold with the isometry group the Lie group $SL(2; \mathbb{R})$. The study of the action of the isometry group on $\mathfrak{H}$ is one of the important ideas we will understand while studying modular forms.

5. **Fundamental Domain:** The idea of a fundamental domain, or fundamental region arises as follows. Given a topological space, and the action of a symmetry group on it, the ***fundamental domain*** is the smallest possible region which can generate the whole space by the action of the group on it. It has one and only one point from each orbit of the group action in its interior. We will give a more complete definition of the fundamental domain of the action of the modular group on the upper half plane, but what we describe here is the intuitive picture that the notion of a fundamental domain attempts to capture. For the theory of modular forms, the space we have in mind is

the upper half plane, $\mathfrak{H}$, and the symmetry groups are $SL(2, \mathbb{Z})$ and its congruence subgroups.

4.2 Towards Modular Functions

From a mathematical point of view, our problem is one of counting. We are interested in counting the partitions of a given entity, say an integer or a vector, as a sum of its constituents taken from some given set S. Restricting at first to just numbers, we ask if a given number can be expressed as a sum of elements of S and, if so, in how many ways can this be done. It is this question that we will chiefly be concerned with in the following chapters, and we will see how the notions we introduce here fit into the idea in a natural way.

Let $p(n)$ denote the number of ways n, an integer, can be written as a sum of elements of S. We ask for the various properties of $p(n)$, say for example, its asymptotic behaviour for large n. We will learn more about the above problem in the course of our study of the Dedekind's eta function and related ideas. For now, we look for a way to motivate the study of modular functions. The partition function $p(n)$ and other functions of additive number theory are intimately related to a class of functions in complex analysis called *elliptic modular functions*. So, it is around this idea that we start our study of modular forms. This chapter is based mostly on [81, 82, 83, 84]

4.2.1 Doubly periodic functions

A function f is said to be an elliptic function if

1. f is doubly periodic.

2. f is meromorphic.

We already know what a meromorphic function is, so we start with the doubly periodic condition. We will see that doubly periodic functions will lead us to the set of lattices in $\mathbb{C}$, and the set of lattices in $\mathbb{C}$ are very closely related to modular forms, which we will come to shortly. On the whole, we will find that elliptic functions, lattices in $\mathbb{C}$, and modular forms are related to each other very closely.

A function f over $\mathbb{C}$ is called *periodic*, with period ω, if

$$f(z + \omega) = f(z) \tag{4.4}$$

whenever z and $z+\omega$ are in the domain of f. An example of such a function would be the exponential function $e^z, z \in \mathbb{C}$ with period $2\pi i$.

A function f is called ***doubly periodic*** if it has two periods ω_1 and ω_2 such that the ratio ω_1/ω_2 is not real.[1] If ω_1 and ω_2 are periods of f, then so is any combination $(m\omega_1 + n\omega_2)$ for any $m, n \in \mathbb{Z}$. The pair (ω_1, ω_2) is called a *fundamental pair*. The set of all linear combinations $m\omega_1 + n\omega_2$ is denoted $\Omega(\omega_1, \omega_2)$. This is called the lattice generated by ω_1 and ω_2. We will see examples of such functions when we consider some of the examples of modular forms later in this chapter.

Let M denote the set of pairs (ω_1, ω_2) of elements of $\mathbb{C}^*$, and $\mathcal{L}$ be the set of all lattices of $\mathbb{C}$. The manifold $\mathbb{C}/L(\omega_1, \omega_2)$ is obtained by identifying the points $z_1, z_2 \in \mathbb{C}$ such that $z_1 - z_2 = \omega_1 m + \omega_2 n$ for some $m, n \in \mathbb{Z}$. Now, given M, the set of all pairs (ω_1, ω_2), we would like to ask when do two such pairs $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$ of M correspond to the same lattice in $\mathcal{L}$? The necessary and sufficient condition for two elements of M to correspond to the same lattice in $\mathcal{L}$ turns out that they should be congruent modlulo $SL(2, \mathbb{Z})$.

The pair (ω_1', ω_2') is equivalent to the pair (ω_1, ω_2) if we can write (ω_1' and ω_2') as

$$\omega_2' = a\omega_2 + b\omega_1 \text{ and } \omega_1' = c\omega_2 + d\omega_1, \tag{4.5}$$

where $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$.

Writing it in a slightly different form leads us to the notion of *unimodular transformations* and the modular group. Let $\tau = \frac{\omega_1}{\omega_2}$, and $\tau' = \frac{\omega_1'}{\omega_2'}$. Then, the above equation in terms of the τ variables is

$$\tau' = \frac{a\tau + b}{c\tau + d}. \tag{4.6}$$

The transformation

$$f(z) = \frac{az + b}{cz + d} \tag{4.7}$$

is called a Möbius transformation. In studying modular forms we will concentrate on such transformations and study functions which are invariant, or have specific transformation properties, under unimodular transformations.

[1] If the ratio of the periods is real and rational, it can be shown that both ω_1 and ω_2 are integer multiples of the same period, and if the ratio is real and irrational, it can be shown that f has arbitrarily small periods and hence is constant on every open connected set on which it is analytic.

4.2.2 Möbius Transformations

The set of Möbius transformations, as defined above, will be important to us when we define the action of the modular group on the upper half plane. So far, we have just defined what are doubly periodic functions, and how the period of the function in two different directions generates a parellogram which defines a lattice as functions of the periods. In seeking to charecterize the distinct pairs of such periods which define the same lattice in $\mathbb{C}$, we came to consider unimodular transformations which relate equivalent sets of pairs or periods. These transformations form a group, as we will see now, but before that we need to extend the domain of definition of the transformations to the extended complex plane $\widetilde{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$ (i.e. $\mathbb{C}$ together with the point at ∞. $\widetilde{\mathbb{C}}$ is also called the Riemann sphere). To do so, we have to extend the definition to the points $z = -\frac{d}{c}$ and $z = \infty$. We define the value of f at these points as follows

$$f(\tfrac{-d}{c}) = \infty \quad \text{and} \quad f(\infty) = \frac{a}{c}, \tag{4.8}$$

with the usual convention that $z/0 = \infty$ if $z \neq$.

A Möbius transformation remains unchanged if we multiply all the coefficients a, b, c, d by the same nonzero constant. Thus, we lose no generality in assuming $ad - bc = 1$. Now, let us associate with each Möbius transformation (4.7), a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{4.9}$$

Also since we have assumed $ad - bc = 1$, $\det A = 1$. Then, the composition of two such transformations, it is easy to verify, is given by the matrix product of the matrices associated to the transformations, and is also a Möbius transformation. The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to the identity transformation

$$f(z) = \frac{1z + 0}{0z + 1} . \tag{4.10}$$

Inverting (4.7), and solving for z in terms of $f(z)$

$$z = \frac{df(z) - b}{-cf(z) + a}, \tag{4.11}$$

shows that f maps $\widetilde{\mathbb{C}}$ to $\widetilde{\mathbb{C}}$. Thus, the inverse of f is also a Möbius transformation and the inverse matrix

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{4.12}$$

corresponds to $f^{-1}(z)$. Thus, we see that the set of all Möbius transformations with $ad-bc = 1$ forms a group. This is no surprise as the matrices A as defined above are just the subgroup of $SL(2,\mathbb{Z})$ with $\det A = 1$. In studying modular forms we study an important subgroup of this group, where all the coefficients a, b, c, d are taken to be integers. It is called the modular group.

4.3 The Modular Group and Fundamental Domain

The set of all Möbius transformations of the form

$$z' = \frac{az+b}{cz+d}, \tag{4.13}$$

with a, b, c, d integers, and $ad - bc = 1$, and each matrix A identified with its negative, $-A$, is called the ***modular group***, denoted $\Gamma(1)$ (The argument 1 becomes clear later, when we discuss congruent subgroups). From its definitions we can see that this is just the group $PSL(2,\mathbb{Z})$ [2]. The group gets its name from the fact that the points of the quotient space $\Gamma(1) \setminus \mathfrak{H}$ are *moduli* for the isomorphism classes of elliptic curves over $\mathbb{C}$. It is the simplest example of a *moduli space*.

Let $\mathfrak{H}$ be the upper half plane. We understand the action of $SL(2,\mathbb{Z})$ on $\widetilde{\mathbb{C}}$ as follows. Let $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ be any element of $SL(2,\mathbb{Z})$ and let $z \in \widetilde{\mathbb{C}}$ be any point in $\widetilde{\mathbb{C}}$. Then, the action of g on z is given by

$$g \cdot z \equiv \frac{az+b}{cz+d}. \tag{4.14}$$

We are representing the transformation by the matrix associated with it. We note the

[2]Some books call the group $SL(2,\mathbb{Z})$ the modular group. We are moding out the center of the group, $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, from it since it acts trivially on $\mathfrak{H}$.

following about the action of $SL(2,\mathbb{Z})$ on $\widetilde{\mathbb{C}}$.

$$\mathrm{Im}(g \cdot z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}, \tag{4.15}$$

showing that the imaginary part of $g \cdot z$ is greater than zero, if the imaginary part of z is. So, $g.z \in \mathfrak{H}$ if $z \in \mathfrak{H}$. Thus, $\mathfrak{H}$ is stable under the action of $SL(2,\mathbb{Z})$. $\Gamma(1)$ is generated by the two elements S and T given by

$$S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.16}$$

with the following relations between them:

$$S^2 = (ST)^3 = \pm 1. \tag{4.17}$$

$\Gamma(1)$ is generated as the free product of the cyclic group of order 2 generated by S and the cyclic group of order 3 generated by ST.

The action of the generators on any $z \in \mathfrak{H}$ is given by

$$S \cdot z = -\frac{1}{z}; \quad T \cdot z = z + 1. \tag{4.18}$$

Two points z and z' in the upper half plane are said to be *equivalent* under $\Gamma(1)$ if $z' = Az$ for some $A \in \Gamma(1)$. Since $\Gamma(1)$ is a group, this is an equivalence relation. This equivalence relation divides the upper half plane into disjoint orbits of the group action and it suffices to consider one point from each orbit to know the action of the whole group on the upper half plane. This set – the union of the representative points of each orbit – is called the *fundamental set* of $\Gamma(1)$. For sets that have a topological structure, it becomes even nicer to consider the topological properties to study the group generating it. This is the notion of a ***fundamental domain***. This is in keeping with our earlier definition of the notion where we said it captures the symmetry structure of the action of a group on a topological space by homeomorphisms. Typically, the fundamental domain always consists of an open set together with a set of few addition points (of measure zero).

For an open set to be the fundamental region of a group it has to have the two following properties

1. No two points of the fundamental domain are equivalent under the group action.

2. For any $z \in \mathfrak{H}$ there is a point z' in the closure of the fundamental domain such that z' is equivalent to z under the group action.

Typically we require it to be connected with some restriction on its boundary. However, it need not necessarily be connected.

The ***fundamental domain*** for the action of the modular group on $\mathfrak{H}$, denoted D, is given by all $z \in \mathfrak{H}$ such that

$$\{|\text{Re}(z)| < \tfrac{1}{2}, |z| > 1\} \cup \{|z| \geq 1, \text{Re}(z) = -\frac{1}{2}\} \cup \{|z| = 1, -\tfrac{1}{2} < \text{Re}(z) < 0\}, \tag{4.19}$$

where the first part is the open set, and the two other terms are the boundaries one on the left, and the other an arc at the bottom, respectively. Let us denote by $O(z) = \{g : g \in \Gamma(1), gz = z\}$ the stabilizer of the point $z \in D$ in $\Gamma(1)$. That is, the set of elements of $\Gamma(1)$ whose action leaves a given element of D invariant. For all the points in D, except the above mentioned three points, the stabilizer is just the identity of $\Gamma(1)$. That is, $O(z) = \{1\}$ for all $z \in D$ except for the following three points

1. $z = i$, in which case $O(z)$ is the group of order 2 generated by S;
2. $z = e^{2\pi i/3}$, in which case $O(z)$ is the group of order 3 generated by ST
3. $z = e^{\pi i/3}$, in which case $O(z)$ is the group of order 3 generated by TS.

With the idea of the modular group, its action on $\mathfrak{H}$, and the fundamental domain, we are now ready to define and study modular functions and modular forms.

4.4 Modular Functions and Modular Forms

Modular functions are meromorphic functions on the upper half plane which are invariant under the modular group. That would correspond to any of the following equivalent objects

1. A function from $\Gamma(1)\backslash\mathfrak{H}$ to $\mathbb{C}$,
2. A function $f : \mathfrak{H} \to \mathbb{C}$ satisfying the transformation equation $f(\gamma z) = f(z)$ for all $z \in \mathfrak{H}$,
3. A function assigning, to every elliptic curve E over $\mathbb{C}$, a complex number depending only on the isomorphism type of E, or

4. A function on lattices in $\mathbb{C}$ satisfying $F(\lambda L) = F(L)$ for all lattices L and all $\lambda \in \mathbb{C}^*$.

Generally, however, the term "modular function" is used only for meromorphic modular functions satisfying certain growth properties. For $k \in \mathbb{Z}$, a function f is said to be ***weakly modular of weight*** k, if f is a meromorphic function on $\mathfrak{H}$ such that for all $g \in \Gamma(1)$ and $z \in \mathfrak{H}$

$$f(z) = (cz + d)^{-k} f(g \cdot z) \ . \tag{4.20}$$

From the above definition we see that the constant functions are modular functions of weight zero. They are invariant under the action of the modular group. The product of two weakly modular functions of weights k_1 and k_2 is a weakly modular function of weight $k_1 + k_2$. There are no modular functions of odd weight. For even k, the above equation is same as

$$f(g \cdot z)(d(g \cdot z))^{k/2} = f(z)(dz)^{k/2} \ . \tag{4.21}$$

In words, the *differential form of weight k*, $f(z)(dz)^{k/2}$, is invariant under the action of $\Gamma(1)$. Since we know that $\Gamma(1)$ is generated by S and T, to know the transformation of a weakly modular function, we just need the transformation properties of the meromorphic function f under S and T. A meromorphic function f on $\mathfrak{H}$ is said to be weakly modular of weight k if it transforms under S and T in the following way:

$$f(-\tfrac{1}{z}) = z^k \ f(z) \tag{4.22}$$

$$f(z + 1) = f(z) \ . \tag{4.23}$$

4.4.1 q-Expansion:

The map $z \mapsto e^{2\pi iz}$ defines a holomorphic map from $\mathfrak{H}$ to the punctured unit disc D' (i.e. open unit disc $|q| < 1$ with the origin removed). Thus, we can Fourier expand $f(z)$ as a function of $q(z) = e^{2\pi iz}$ as $f(z) = \sum_{-N}^{\infty} a_n q^n$. Then, since $f(z+1) = f(z)$, consider the space $\mathfrak{H}/T$, that is the quotient space of $\mathfrak{H}$ modulo translation by integers (a cylinder). q induces an isomorphism between $\mathfrak{H}/T$ and the punctured disc. Thus, a meromorphic function f on $\mathfrak{H}$ which satisfies the condition (4.23) above (invariance under T), induces a meromorphic function, f_∞, on the punctured disc such that $f_\infty(q(z)) = f(z)$. If the meromorphicity (holomorphicity) of f_∞ extends to 0, we say that f is *meromorphic (holomorphic) at infinity*. A necessary and sufficient condition that f_∞ is also meromorphic at 0 is that there exists some positive integer N such that $f_\infty(q)q^N$ is bounded near 0. f_∞ then admits a Laurent expansion in the neighborhood of the origin.

$$f_\infty(q) = \sum_{-N}^{\infty} a_n q^n \ . \tag{4.24}$$

The above is called the *q-expansion* of f about ∞. The coefficients a_n are the Fourier coefficients of f.

Definition. A weakly modular function of weight k is called a ***modular function*** if it is meromorphic at ∞.

If f is holomorphic at ∞, we set $f(\infty) = f_\infty(0)$. This is the value of f at ∞.

Definition. A modular function which is holomorphic everywhere on $\mathfrak{H}$ and at ∞ is called a ***modular form*** of weight k (and level 1).

If f is a modular form, then there are numbers a_n such that for all $z \in \mathfrak{H}$, f is given by a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \tag{4.25}$$

which converges for $|q| < 1$ (i.e. $z \in \mathfrak{H}$). A modular form of weight k is called a ***cusp form of weight*** k (and level 1) if $f(\infty) = 0$, i.e., $a_0 = 0$. We will use the following notation for the action of the modular group on f.

$$f[\alpha]_k = f(\alpha z)(cz + d)^{-k}(\det\alpha)^{k/2} \ . \tag{4.26}$$

4.5 Congruence Subgroups

As the name suggests, congruence subgroups, of a matrix group, are subgroups defined by congruence condition on the entries of the matrix. The matrix group we are interested in is $PSL(2,\mathbb{Z})$. The congruence subgroups of $PSL(2,\mathbb{Z})$ arise in the following way. Given the group $PSL(2,\mathbb{Z})$, we can restrict the entries to be in $\mathbb{Z}/N\mathbb{Z}$, obtaining the homomorphism

$$PSL(2,\mathbb{Z}) \to PSL(2;\mathbb{Z}/N\mathbb{Z}) \tag{4.27}$$

between the two groups. The kernel (i.e. the inverse image of the identity e) of this map is an example of a congruence subgroup and is called ***the principal congruence subgroup of level n***,$\Gamma(N)$. It is given by $a \equiv d \equiv \pm 1$, $b \equiv c \equiv 0$ (mod N) (Now we see where the

1 in $\Gamma(1)$ for the full modular group comes from). $\Gamma(N)$ is, in fact, a normal subgroup of $\Gamma(1)$, as can be easily verified by seeing that $A^{-1}BA \in \Gamma(1)$ for any two matrices $A \in \Gamma(1)$ and $B \in \Gamma(N)$. The ***index*** of $\Gamma(N)$ in $\Gamma(1)$ is the number of equivalence classes of matrices modulo N. We can take the inverse image of any subgroup (not just the identity e) and that gives other congruence subgroups. The ones we will be studying in relation to counting of BPS states are the following subgroups of $\Gamma(1)$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\bmod N) \right\} \tag{4.28}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\bmod N) \right\}, \tag{4.29}$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} (\bmod N) \right\}, \tag{4.30}$$

where * means any element. The number N is called the ***level*** of Γ. We can define modular functions for the congruence subgroups just as in the case of the full modular group.

4.6 Lattices

We earlier said one of the ways in which the modular group arises is by considering the set of all lattices in $\mathbb{C}$. Lattices in $\mathbb{C}$ are closely related to modular forms defined above. We will see that, upto certain transformations, the quotient $\mathfrak{H}/\Gamma(1)$ can be identified with a lattice of $\mathbb{C}$. Most of the discussion will not be too formal, but we give the formal definitions of key ideas for the sake of completeness. We first define what we mean by a lattice in the real vector space $\mathbb{V}$. There are several ways of defining a lattice in a vector space, we give one that is easiest to understand below.

A ***lattice*** in a real vector space $\mathbb{V}$ of finite dimension is a discrete subgroup, L, of $\mathbb{V}$ such that $\mathbb{V}/L$ is compact. Similarly, one can define a lattice in a complex vector space. Specifically, consider $\mathbb{C}$. Given two non-vanishing complex numbers ω_1 and ω_2 such that $\omega_1/\omega_2 \notin \mathbb{R}$, we can associate a lattice, $L(\omega_1, \omega_2)$, to ω_1 and ω_2 by $L(\omega_1, \omega_2) \equiv \{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\}$. We assume $\text{Im}(\omega_2/\omega_1) > 0$. $\{\omega_1, \omega_2\}$ is the basis of L. Let M denote the set of pairs (ω_1, ω_2) of elements of $\mathbb{C}^*$, and let $\mathcal{L}$ be the set of all lattices of $\mathbb{C}$. The manifold $\mathbb{C}/L(\omega_1, \omega_2)$ is obtained by identifying the points $z_1, z_2 \in \mathbb{C}$ such that $z_1 - z_2 = \omega_1 m + \omega_2 n$ for some $m, n \in \mathbb{Z}$.

Now, given M, the set of all pairs (ω_1, ω_2), we would like to ask when do two such pairs $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$ of M correspond to the same lattice in $\mathcal{L}$? The necessary and sufficient condition for two elements of M to correspond to the same lattice in $\mathcal{L}$ turns out that they should be congruent modlulo $SL(2, \mathbb{Z})$. Thus, we see that we can identify the set $\mathcal{L}$ of lattices of $\mathbb{C}$ with the quotient of M by the action of $SL(2, \mathbb{Z})$.

Also, since it is only the ratio that determines the lattice, we can act by $\mathbb{C}^*$ on any element (ω_1, ω_2) of M (respectively $\mathcal{L}$) as follows

$$(\omega_1, \omega_2) \mapsto (\lambda\omega_1, \lambda\omega_2), \quad (\text{ resp. } L \mapsto \lambda L), \quad \lambda \in \mathbb{C}^*, \tag{4.31}$$

without changing the ratio. Thus, we can identify the quotient $M/\mathbb{C}^*$ with $\mathfrak{H}$ by $(\omega_1, \omega_2) \mapsto z = \omega_1/\omega_2$, and thus, this identification transforms the action of $SL(2, \mathbb{Z})$ on M into that of $\Gamma(1)$ on $\mathfrak{H}$. We make this idea precice below, where we explain what we said in the beginning about the identification of a lattice of $\mathbb{C}$ with the quotient $\mathfrak{H}/\Gamma(1)$. The map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ gives a bijection of $\mathcal{L}/\mathbb{C}^*$ onto $\Gamma(1)\backslash\mathfrak{H}$. Thus, we can identify an element of $\Gamma(1)\backslash\mathfrak{H}$ with a lattice of $\mathbb{C}$ upto a homothety (dilation).

For $k \in \mathbb{Z}$, we say that a complex valued function, F, on $\mathcal{L}$ is of weight k if

$$F(\lambda L) = \lambda^{-k} F(L) \tag{4.32}$$

for all lattices $L \in \mathcal{L}$ and all $\lambda \in \mathbb{C}^*$. Let us denote by $F(\omega_1, \omega_2)$ the value of F on the lattice $L(\omega_1, \omega_2)$. Then the above formula is just

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} F(\omega_1, \omega_2)\ . \tag{4.33}$$

We note in the above formula that the product $\omega_2^{-k} F(\omega_1, \omega_2)$ depends only on $z = \omega_1/\omega_2$. Thus, we can always find a function, f, on $\mathfrak{H}$ such that

$$F(\omega_1, \omega_2) = \omega_2^{-k} f(\omega_1/\omega_2) \tag{4.34}$$

Also, since $F(\omega_1, \omega_2)$ is invariant under an $SL(2, \mathbb{Z})$ action on M, we see that f satisfies the following identity:

$$f(z) = (cz+d)^{-k} f(\frac{az+b}{cz+d}) \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.35}$$

Conversely, if f verifies the above formula, then we can associates it to a function F on $\mathcal{L}$ which is of weight k. We thus get a correspondence between *modular functions of weight k* and *lattice functions of weight k*.

4.7 Examples of Modular Forms

We are now ready to see some examples of modular forms and all the theory we learnt being put to use. We will learn such examples that we will have occasion to use later in studying the main problem of this thesis. We will look at the following examples:

1. Eisenstein series, which will be used in constructing the twisted elliptic genera of the $K3$ manifold.
2. Seigel modular forms, which give the degeneracy of the dyons in certain models of string theory that we will consider, and are at the heart of this thesis.
3. Jacobi forms - the theta functions, and the Fourier coefficients of the Siegel modular forms considered above.

There are many more important and illustrative examples of modular forms like the J function, the Δ function (which occurred as the generating function of the multiplicities of the roots of the fake monster algebra (3.124)), Weierstrass $\wp$ function, and many more, but we will not discuss them here. The above three examples are not only very important examples of modular forms, but they also play a very important role in the construction of the dyon degeneracy partition function. Of the three, we will spend considerable time on Siegel modular forms given their importance from the point of view of this work. We start with the Eisenstein series.

4.7.1 Eisenstein Series:

Let L be a lattice in $\mathbb{C}$. Consider the series $\sum_{\gamma\in L} 1/|\gamma|^{\sigma}$. This series is convergent for $\sigma > 2$, where the $\sum$ runs over the nonzero elements of Γ. Thus, the series

$$G_{2k}(L) = \sum_{\gamma\in\Gamma} 1/\gamma^{2k} \tag{4.36}$$

will be absolutely convergent for any integer $k > 1$. It is called the (non-normalized) ***Eisenstein series of index*** $2k$. Writing G_{2k} as a function on M we get

$$G_{2k}(\omega_1, \omega_2) = \sum_{m,n}{}' \frac{1}{(m\omega_1 + n\omega_2)^{2k}} \tag{4.37}$$

where the summation is over all pairs of integers $(m, n) \neq (0, 0)$ which we indicate by a prime in the superscript. From the preceding section, the function on $\mathfrak{H}$ corresponding to $G_{2k}(\omega_1, \omega_2)$ is given by

$$G_{2k}(z) = \sum_{m,n} \frac{1}{(mz+n)^{2k}} \tag{4.38}$$

where again the summation is over pairs of integers m, n such that $(m, n) \neq (0, 0)$. Let us see how the T and S transformations act on the form $G_k(z)$. Under a T transformation, $z \mapsto z + 1$, therefore $G_{2k}(z) \mapsto G_{2k}(z + 1)$ as follows

$$G_{2k}(z+1) = \sum_{m,n}{}' \frac{1}{(m(z+1)+n)^{2k}} = \sum_{m,n}{}' \frac{1}{(mz+(n+m))^{2k}} = \sum_{m,n}{}' \frac{1}{(mz+n')^{2k}} = G_{2k}(z). \tag{4.39}$$

Under an S transformation $z \mapsto -\frac{1}{z}$, thus $G_{2k}(z) \mapsto G_{2k}(-\frac{1}{z})$ as follows

$$G_{2k}(-\tfrac{1}{z}) = \sum_{m,n} \frac{1}{(-m/z+n)^{2k}} = \sum_{m,n} \frac{z^2k}{(-m+nz)^{2k}} = z^2k \sum_{m,n} \frac{1}{(m'z+n')^{2k}} = z^k G_{2k}(z). \tag{4.40}$$

Thus, we see that $G_{2k}(z)$ (and hence, $G_{2k}(L)$ and $G_{2k}(\omega_1, \omega_2)$) is a modular form of weight $2k$ with the value at ∞ given by $G_{2k}(\infty) = 2\zeta(2k)$, where ζ is the Riemann zeta function. Often, the Eisenstein series is redefined, so that the constant term is 1, by dividing it by

$2\zeta(2k)$. This is called the ***normalized Eisenstein series***

$$E_{2k}(z) = \frac{G_{2k}(z)}{2\zeta(2k)}. \tag{4.41}$$

As before, we can also consider Eisenstein series with respect to a subgroup $\Gamma(N)$ of $\Gamma(1)$, instead of the whole modular group. This gives Eisenstein series at level N. Below we give some explicit expansions of some of the Eisenstein series at various levels.

4.7.2 Fourier Expansions of Eisenstein Series:

As a modular form the Eisenstein series will admit a q-expansion as formal power series in terms of $q(z) = e^{2\pi i\tau}$. Here we give the Fourier expansion of the Eisenstein series $G_k(z)$ in terms of *Bernoulli numbers* B_n and the *sigma function* which we define below.

Definition 4.7.1 *Sigma Function: For an integer $r \geq 0$ and any positive integer n, the sigma function is defined as the sum of the r-th powers of the positive divisors of n. i.e.*

$$\sigma_r(n) = \sum_{1\leq d|n} d^r. \tag{4.42}$$

We also set $\sigma_0(n) = d(n)$ for the number of positive divisors of n and $\sigma(n) = \sigma_1(n)$.

Definition 4.7.2 *Bernoulli Numbers: For $n \neq 0$ the Bernoulli numbers, B_n, are defined by the following equality of formal power series:*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{4.43}$$

Using the above two definitions, we can write the Fourier expansion of the normalized Eisenstein series E_{2k} as [3]

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n. \tag{4.44}$$

For Eisenstein series of higher level, we have to compute their explicit form using various relations between the E_{2k}s. A discussion of Eisenstein series at level N and their expansions about different cusps is discussed in Appendix B. Below, as an example, we give the Fourier

[3] E_2 as defined below is **not** a modular forms of weight two due to convergence. A closely related non-holomorphic form $E_2^* = \left(E_2 - \frac{3}{\mathrm{Im}\, z}\right)$ has weight two (See Appendix B)

expansion for $E_2^{(2)}, E_2^{(3)}, E_2^{(4)}$ and $E_2^{(5)}$.

$$E_2^{(2)}(\tau) = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + 96q^6 + 192q^7 + 24q^8 + 312q^9 + \cdots$$
$$E_2^{(3)}(\tau) = 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + 36q^6 + 96q^7 + 180q^8 + 12q^9 + \cdots$$
$$E_2^{(4)}(\tau) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + \cdots \quad (4.45)$$
$$E_2^{(5)}(\tau) = 1 + 6q + 18q^2 + 24q^3 + 42q^4 + 6q^5 + 72q^6 + 48q^7 + 90q^8 + 78q^9 + \cdots \quad (4.46)$$

The Eisenstein series are very important in the theory of automorphic forms and occur in a number of places. Here we have listed only the very basic facts about them and the interested reader is refered to any of the references for a more complete discussion.

4.7.3 Jacobi Forms

In this section we study another important example – that of Jacobi forms. We will study two examples of Jacobi forms - the Theta series in this section, and the Fourier-Jacobi development of Siegel modular forms when we study Siegel modular forms in the next section. Jacobi forms are a cross between elliptic functions and modular forms in one variable in that one of the variable it takes is from $\mathbb{C}$, while the other is restricted to $\mathfrak{H}$.

A ***Jacobi form*** on $SL(2,\mathbb{Z})$ is a holomorphic function

$$\phi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C} \quad (4.47)$$

satisfying the two transformation equations

$$\phi\Big(\frac{az+b}{cz+d}, \frac{\tau}{cz+d}\Big) = (cz+d)^{2k} e^{\frac{2\pi imc\tau}{cz+d}\phi(z,\tau)} \quad (4.48)$$

$$\phi(z, \tau + \lambda z + \mu) = e^{-2\pi im(\lambda^2 z + 2\lambda\tau)}\phi(z,\tau) \quad \lambda, \mu \in \mathbb{Z}^2) \ . \quad (4.49)$$

These two sets of transformations define the Jacobi group (See Appendix D). $\phi(z,\tau)$ has a Fourier expansion of the form

$$\phi(z,\tau) = \sum_{n=0}^{\infty} \sum_{r\in\mathbb{Z}, r^2 \leq 4mn} c(n,r) e^{2\pi i(nz + r\tau)} \quad (4.50)$$

where $k, m \in \mathbb{N}$ are called the *weight* and *index* of ϕ, respectively, and the Fourier coefficients, $c(n,r) = 0$, unless $n \geq 0$ and $4mn - r^2 \geq 0$. Note that the function $\phi(z, 0)$ is an ordinary modular form of weight $2k$. If $m = 0$, then ϕ is independent of τ and the definition reduces to the usual notion of modular forms in one variable.

For *weak* Jacobi forms, the coefficients $c(n, r)$ are non-vanishing only when $n \geq 0$ relaxing the condition involving $(4nt - \ell^2)$. Jacobi forms of integer index were considered by Eichler and Zagier[83] and extended to half-integral indices by Gritsenko [85].

The elliptic genus of Calabi-Yau manifolds are weak Jacobi forms. Examples include:

$$\begin{aligned} \phi_{-2,1}(z_1, z_2) &= \mathcal{E}_{\mathrm{st}\times T^2}(z_1, z_2) = \left(\frac{i\vartheta_1(z_1, z_2)}{\eta^3(z_1)}\right)^2 \\ \phi_{0,1}(z_1, z_2) &= \mathcal{E}_{K3}(z_1, z_2) = 8\sum_{i=2}^{4}\left(\frac{\vartheta_i(z_1, z_2)}{\vartheta_i(z_1, 0)}\right)^2 \end{aligned} \tag{4.51}$$

We will see the appearance of weight zero Jacobi forms of the group $\Gamma_0(N)^J$ in writing product representations for the modular form $\Phi_k(\mathbf{Z})$.

$$\phi_{0,1}^{(N)}(\tau, z) = \frac{2N}{N+1}\ \alpha^{(N)}(\tau)\ \phi_{-2,1}(\tau, z) + \frac{1}{N+1}\ \phi_{0,1}(\tau, z)\ , \tag{4.52}$$

with $\alpha^{(N)}(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau\big[\ln\eta(\tau) - \ln\eta(N\tau)\big]$ is the Eisenstein series for $\Gamma_0(N)$. The Fourier expansion for $\phi_{0,1}^{(N)}$ at the cusp at $i\infty$

$$\begin{aligned} \phi_{0,1}^{(2)}(\tau, z) &= \left(2r + 4 + \frac{2}{r}\right) + \left(4r^2 - 8 + \frac{4}{r^2}\right) q + O\left(q^2\right) \\ \phi_{0,1}^{(3)}(\tau, z) &= \left(2r + 2 + \frac{2}{r}\right) + \left(2r^2 - 2r - \frac{2}{r} + \frac{2}{r^2}\right) q + O\left(q^2\right) \\ \phi_{0,1}^{(5)}(\tau, z) &= \left(2r + \frac{2}{r}\right) + \left(2r - 4 + \frac{2}{r}\right) q + O\left(q^2\right). \end{aligned} \tag{4.53}$$

and about the cusp at 0 is

$$
\begin{aligned}
\phi_{0,1}^{(2)} =&8+\left(-\frac{16}{r}+32-16r\right)q^{1/2}+\left(\frac{8}{r^2}-\frac{64}{r}+112-64r+8r^2\right)q+O\left(q^{3/2}\right)\\
\phi_{0,1}^{(3)} =&6+\left(-\frac{6}{r}+12-6r\right)q^{1/3}+\left(-\frac{18}{r}+36-18r\right)q^{2/3}\\
&+\left(\frac{6}{r^2}-\frac{42}{r}+72-42r+6r^2\right)q+O\left(q^{4/3}\right)\\
\phi_{0,1}^{(5)} =&4+\left(-\frac{2}{r}+4-2r\right)q^{1/3}+\left(-\frac{6}{r}+12-6r\right)q^{2/5}+\left(-\frac{8}{r}+16-8r\right)q^{3/5}\\
&+\left(-\frac{14}{r}+28-14r\right)q^{4/5}+\left(\frac{4}{r^2}-\frac{26}{r}+44-26r+4r^2\right)q+O\left(q^{6/5}\right).
\end{aligned} \tag{4.54}
$$

Theta Functions

The ***Jacobi theta function*** is a function of two variable τ and z, where $\tau \in \mathfrak{H}$, $\text{Im}\tau > 0$ and $z\ in\mathbb{C}$ defined by

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{(\pi i n^2\tau + 2\pi i n z)} . \tag{4.55}$$

One can look at it as a Fourier series for a function in z which is periodic with respect to $z \mapsto z+1$ by writing it as

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} a_n(\tau)e^{2\pi i n z)}, \qquad \text{where } a_n(\tau) = e^{\pi i n^2\tau} \tag{4.56}$$

from where the $\vartheta(z,\tau) = \vartheta(z+1,\tau)$ part is obvious.

4.8 Siegel Modular Forms

In studying Siegel modular forms we will generalize elliptic modular forms on $SL(2,\mathbb{Z})$ to a more general class of modular forms known as vector valued modular forms. Viewed from this point of view it becomes easier to motivate intuitively the construction of Siegel modular forms along the lines of elliptic modular forms by suitably generalizing each notion involved in the definition. The modular forms we have studied so far are holomorphic maps from the complex upper half plane $\mathfrak{H}$ to $\mathbb{C}$. For more general contexts, we would like to study modular forms more general than ones with values in $\mathbb{C}$. Vector valued modular forms map the Siegel upper half plane (a generalization of the complex upper half plane) to a vector space V. We define all the relevant ideas as we go along and put together the definition of Siegel modular forms, but before that we recollect some basic definitions.

The symplectic group plays an important role in the theory of Siegel modular forms and we start by recalling its definition.

Definition 4.8.1 *Symplectic Matrix: A $2g \times 2g$ matrix M is said to be a symplectic matrix if it satisfies the following condition*

$$M^T \Omega M = \Omega, \tag{4.57}$$

where M^T denotes the transpose of M and Ω is the fixed nonsingular, skew-symmetric block matrix generally taken to be

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{4.58}$$

where I_g is the $g \times g$ identity matrix.

The above condition on symplectic matrices can also be expressed equivalently as follows. Let the $2g \times 2g$ matrix M be a block matrix given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.59}$$

where each of A, B, C and D are $g \times g$ matrices. Then the above condition is equivivalent to

$$AB^T = BA^T, \quad CD^T = DC^T, \quad \text{and } AD^T - BC^T = 1_g. \tag{4.60}$$

There is more than one way of expressing the above relations and any one suffices. Ω has determinant $+1$ and its inverse is given by $\Omega^{-1} = \Omega^T = -\Omega$..

Every symplectic matrix is invertible with the inverse given by

$$M^{-1} = \Omega^{-1} M^T \Omega. \tag{4.61}$$

Further, the product of two symplectic matrices is, again, a symplectic matrix. Thus, we see that the set of all symplectic matrices has the structure of a group. This group is known as the symplectic group.

Definition 4.8.2 *Symplectic group: The symplectic group of degree $2g$ over a field $\mathbb{F}$, denoted $Sp(g, \mathbb{F})$, is the group of $2g \times 2g$ matrices with entries in $\mathbb{F}$, and with the group operation* *as matrix multiplication.*

More generally, it is the set of linear transformations of a $2g$-dimensional symplectic vector space (a vector space with a nondegenerate, skew-symmetric bilinear form known as the symplectic form) over F. For our purposes, we will only be working with $Sp(g, \mathbb{Z})$. Since every symplectic matrix has determinant $+1$, $Sp(2, \mathbb{Z})$ is a subgroup of $SL(2, \mathbb{Z})$ and is a discrete subgroup of $Sp(g, \mathbb{R})$ just as $SL(2, \mathbb{Z})$ is of $SL(2, \mathbb{R})$.

We now start our study of Siegel modular forms. We said they generalize the notion of ordinary modular forms to vector valued modular forms, so let us understand their construction by generalizing ordinary modular forms. To define an elliptic modular form we needed the concept of a holomorphic function on $\mathbb{C}$, the upper half plane $\mathfrak{H}$, the group $SL(2, \mathbb{Z})$ and its action on $\mathfrak{H}$ (or rather, of the quotient, the modular group $\Gamma(1)$) and the factor of automorphy $(cz + d)^k$. To generalize the definition to vector valued modular forms, we need to suitably generalize each of the notions in the definition.

4.8.1 The group.

The group $SL(2, \mathbb{Z})$ is the automorphism group of the $\mathbb{Z}_2$ lattice with the standard alternating form[4] $\langle , \rangle$ with

$$\langle (a, b), (c, d) \rangle = ad - bc. \tag{4.62}$$

We consider a more general lattice $\mathbb{Z}_{2g}$ [5] of rank $2g$, $g \in \mathbb{Z}_{\geq 1}$, equiped with a symplectic form $\langle , \rangle$ acting on the basis vectors $e_i, \ldots, e_g, f_1, \ldots, f_g$ as follows

$$\langle e_i, e_j \rangle = 0, \quad \langle f_i, f_j \rangle = 0, \quad \textit{and} \quad \langle e_i, f_j \rangle = \delta_{ij}, \tag{4.63}$$

with δ_{ij} is the Kronecker delta. From the definition of a symplectic group above, the automorphism group of this lattice will be the symplectic group $Sp(g, \mathbb{Z})$. In the present context it is called the ***Siegel modular group*** often denoted Γ_g. Thus, the generalization of the modular group, for ordinary modular forms, is the Siegel modular group.

[4]An alternating form is a bilinear form B on a vector space V such that for all $v \in V$, $B(v, v) = 0$. By this property it is automatically skew-symmetric, as it should be for a symplectic vector space.

[5]for finite-dimensional symplectic vector spaces, the dimension is necessarily even since the determinant of an odd dimensional skew-symmetric matrix vanishes.

4.8.2 The Upper Half Space.

Next we have to accordingly generalize the upper half space, on which the modular group acts, to a suitable space on which the Siegel modular group acts. Modular functions were linear transformations, with certain prescribed transformation properties, of the complex upper half plane, which consists of elements in the complex plane with positive definite imaginary part. Since now we are looking at linear transformations of $\mathbb{Z}_{2g}$, the space we are looking for will be a space of matrices. The apropriate generalization to the upper half space, known as the ***Siegel upper half space*** is the set of $g \times g$ complex symmetric matrices with a positive definite imaginary part (obtained by taking the imaginary part of every individual matrix entry).

Definition 4.8.3 *The Siegel upper half space, denoted $\mathfrak{H}_g$ is defined as*

$$\mathfrak{H}_g = \{\tau \in M_{\mathbb{C}}(g,g) : \tau^t = \tau, Im(\tau) > 0\}, \tag{4.64}$$

where $M_{\mathbb{C}}(g,g)$ is the set of $g \times g$ matrices over $\mathbb{C}$.

Justifying the word 'generalization', we get back $\mathfrak{H}$ as $\mathfrak{H}_1$ when $g = 1$.

We must now define the action of the Siegel modular group on $\mathfrak{H}_g$, which is done as follows. The action of $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z})$ on $\tau \in \mathfrak{H}_g$ is given by

$$\tau \mapsto \gamma(\tau) = (A\tau + B)(C\tau + D)^{-1}. \tag{4.65}$$

This action is well defined, in particular $(C\tau + D)$ is invertible, and $\gamma(\tau)$ is symmetric. Also the imaginary part of the transformed matrix, $\mathrm{Im}(\gamma(\tau))$, is positive definite, as it should be, and is again in $\mathfrak{H}_g$.

Given this action, it is natural, as before, to look for the fundamental domain for the action of the group on Γ_g. Siegel constructed a fundamental domain for $g \geq 2$ but they are not as easy to work with as was with the case of ordinary modular forms, and are of limited help in understanding the group action. We will not have to say much about fundamental domains in this section.

4.8.3 The Automorphy Factor

We now need to only generalize the automorphy factor $(C\tau + D)^k$ to the case of Siegel modular forms. This can be easily done noting that C and D are matrices and we know the modular form takes values in a vector space, say V, so we need it to be a map from a space of matrices to a vector space. Thus, we need to consider a representation of $GL(g,\mathbb{C})$ in V. Consider the representation

$$\rho : GL(g,\mathbb{C}) \to GL(V) \tag{4.66}$$

where V is a finite-dimensional vector space over $\mathbb{C}$ provided with a hermitian metric.

Now, we are ready to define a Siegel modular form.

Definition 4.8.4 *Siegel Modular Form of Weight ρ A holomorphic map $f : \mathfrak{H}_g \to V$ is called a* ***Siegel modular form of weight*** *ρ if*

$$f(\gamma(\tau)) = \rho(C\tau + D)f(\tau) \tag{4.67}$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z})$ and all $\tau \in \mathfrak{H}_g$. For $g = 1$, we require that f is holomorphic at ∞.

Modular forms of weight ρ form a $\mathbb{C}$- vector space $M_\rho = M_\rho(\Gamma_g)$, and all the M_ρ are finite dimensional. If ρ is a direct sum of two representations $\rho = \rho_1 \oplus \rho_2$, then $M\rho$ is isomorphic to the direct sum $M_{\rho_1} \oplus M_{\rho_2}$ and so we can restrict ourselves to considering M_ρ for only the irreducible representations of $GL(g,\mathbb{C})$.

We also define scalar-valued Seigel modular forms of weight k, known as ***classical Seigel modular forms***, below.

Definition 4.8.5 *Classical Siegel modular form: A classical Siegel modular form of weight k and degree g is a holomorphic function $f : \mathfrak{H}_g \to \mathbb{C}$ such that*

$$f(\gamma(\tau)) = \det(c\tau + d)^k f(\tau) \tag{4.68}$$

for all $\gamma = (a, b; c, d) \in Sp(g,\mathbb{Z})$ (with for $g = 1$ the usual holomorphicity requirements at ∞).

We denote by $M_k = M_k(\Gamma_g)$ the vector space of classical Siegel modular forms of weight k. These spaces form a graded ring $M^{cl} := \oplus M_k$ of M of classical Siegel modular forms. When $g = 1$, this simply reduces to the usual modular forms on $SL(2,\mathbb{Z})$.

4.8.4 Fourier Expansions

Analogous to the elliptic modular forms on $SL(2,\mathbb{Z})$, we can expand the vector valued modular forms in a Fourier series. In fact, the Siegel modular forms can be constructed and expressed in more than one ways. We will study two of these, using the theta series, and the Fourier-Jacobi development, since not only are both important ways of constructing Siegel modular forms in general, but both constructions are important to us particularly for our study of the Siegel modular forms occuring in counting $\frac{1}{4}$-BPS states in string theory. In constructing the partition function for the degeneracy of $\frac{1}{4}$-BPS states in the string models we are interested in, we will make use of both the approaches. Below we discuss the q-expansion of Siegel modular forms before studying the above mentioned expansions.

For every symmetric $g \times g$ matrix $n \in GL(g, \mathbf{Q})$, such that $2n$ is an integral matrix, we can define a linear form with integral coefficients in the coordinates τ_{ij} of the Siegel upper half space $\mathfrak{H}_g$ as follows

$$\mathrm{Tr}(n\tau) = \sum_{i=1}^{g} n_{ii}\tau_{ii} + 2 \sum_{1\leq i\leq j\leq g} n_{ij}\tau_{ij} \ . \tag{4.69}$$

Also, every integral combination of the coordinates is of this form. The matrix n is called a ***half-integral*** matrix. Now, a function $f : \mathfrak{H}_g \to \mathbb{C}$ that is periodic in the sense that $f(\tau + s) = f(\tau)$ for all symmetrix $g \times g$ matrices s admits a Fourier expansion

$$f(\tau) = \sum_{n \ \text{half integral}} a(n) e^{2\pi i \mathrm{Tr}(n\tau)} \tag{4.70}$$

with $a(n) \in \mathbb{C}$ given by the Fourier transform of $f(\tau)$ as

$$a(n) = \int_{x \bmod 1} f(\tau) e^{-2\pi i \mathrm{Tr}(n\tau)} dx, \tag{4.71}$$

where dx is the Euclidean volume of the space of x-coordinates and the integral runs over $-\frac{1}{2} \leq x_{ij} \leq \frac{1}{2}$. This series is uniformly convergent on compact subsets.

For the case of vector-valued modular forms in in M_ρ we have a similiar Fourier series where the coefficients $a(n)$ will take values in the vector space instead of $\mathbb{C}$ as in the case of periodic functions defined above and satisfy

$$a(u^{\mathrm{T}} n u) = \rho(u^{\mathrm{T}}) a(n) \quad \text{for all } u \in GL(g, \mathbb{Z}) \ . \tag{4.72}$$

Like before we write $q^n = e^{2\pi i \mathrm{Tr}(n\tau)}$ and write (4.70) as

$$f(\tau) = \sum_{n \text{half-integral}} a(n) q^n \ . \tag{4.73}$$

A modular form $f = \sum_n a(n) e^{2\pi i \mathrm{Tr}(nr)} \in M_k(\Gamma_g)$ is called singular if $a(n) \neq 0$ implies that n is a singular matrix (i.e. $\det(n) = 0$).

With this general introduction on the Fourier expansion of a Siegel modular form, along the lines of the $g = 1$ case, we move on to study other developments that exist for $g > 1$ that provide more information about the Siegel modular forms.

4.8.5 The Fourier-Jacobi development of a Siegel modular form.

For $g = 1$, we saw there exists a Fourier expansion for the Siegel modular forms. For $g > 1$, there are other developments of Siegel modular forms which are more general than the Fourier expansion. We will examine the Fourier-Jacobi epansion of a Siegel modular form here. Though the Fourier-Jacobi development is valid and extremely useful for any general $g > 1$, we will in keeping with the scope of this work, restrict ourselves to the case of $g = 2$.

Consider (4.70), where the function $f(\tau) : \mathfrak{H}_g \to \mathbb{C}$, which is periodic in the parameter τ, is expanded as a Fourier series in terms of the $e^{2\pi i \mathrm{Tr}(n\tau)}$ with the coefficients $a(n) \in \mathbb{C}$. Here $\tau \in \mathfrak{H}_g$. Now, suppose we wanted to isolate the periodicity of a Siegel modular form $f(\tau)$, of weight k on Γ^g, under $\tau' \in \mathfrak{H}_1$, as against $\tau \in \mathfrak{H}_g$, and Fourier expand $f(\tau)$ in terms of $e^{2\pi i \mathrm{Tr}(n\tau')}$. The analog of the coefficients $a(n)$ would now correspond to functions which take values from $\mathfrak{H}_{g-1}$. Specializing to our case of $g = 2$, we can Fourier expand the Siegel modular form in terms of one of the variable and obtain what is called the ***Fourier-Jacobi*** development of the Siegel modular form. Let us write the matrix $\mathbf{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2$ (notation in keeping with the additive lifts of Siegel modular forms to be studied in Chatter 5), then the Fourier expansion can be written as

$$f(\mathbf{Z}) = \sum_{m=0}^{\infty} \phi_m(z_1, z_2) e^{2\pi i m z_3} \ . \tag{4.74}$$

where the function $\phi_m(z_1, z_2)$ is now a Jacobi form of weight k and index m (recall $f(\tau)$ was a Siegel modular form of weight k). This means ϕ_m satisfies

1. $\phi_m((az_1+b)/(cz_1+d), z_2/(cz_1+d)) = (cz_1+d)^k e^{2\pi imcz_2^2/(cz_1+d)}\phi_m(z_1,z_2)$,

2. $\phi_m(z_1, z_2+\lambda z_1+\mu) = e^{-2\pi im(\lambda^2 z_1+2\lambda z_2)}\phi_m(z_1,z_2)$

3. ϕ_m has a Fourier expansion of the form

$$\phi_m = \sum_{n=0}^{\infty}\sum_{r\in\mathbb{Z}, r^2\leq 4mn} c(n,r)e^{2\pi(nz_1+rz_2)} \ . \qquad (4.75)$$

This gives a relation between Siegel modular forms for genus 2 and Jacobi forms and we will use this correspondence later in deriving the degeneracy of $\frac{1}{4}$-BPS states from the degeneracy of $\frac{1}{2}$-BPS states.

4.8.6 Theta Series

We define the genus-two theta constants as follows[86]:

$$\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{Z}) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} q^{\frac{1}{2}(l_1+\frac{a_1}{2})^2}\ r^{(l_1+\frac{a_1}{2})(l_2+\frac{a_2}{2})}\ s^{\frac{1}{2}(l_2+\frac{a_2}{2})^2}\ e^{i\pi(l_1b_1+l_2b_2)}\ , \qquad (4.76)$$

where $\mathbf{a} = \begin{pmatrix}a_1\\ a_2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}b_1\\ b_2\end{pmatrix}$, and $\mathbf{Z} = \begin{pmatrix}z_1 & z_2\\ z_2 & z_3\end{pmatrix} \in \mathbb{H}_2$. Further, we have defined $q = \exp(2\pi iz_1)$, $r = \exp(2\pi iz_2)$ and $s = \exp(2\pi iz_3)$. The constants (a_1,a_2,b_1,b_2) take values $(0,1)$. For even$\mathbf{a}^{\mathrm{T}}\mathbf{b}$, it yields the so called ***even theta constants***. Thus there are sixteen genus-two theta constants. There are ten such theta constants for which we list the values of $\mathbf{a}$ and $\mathbf{b}$:

m	0	1	2	3	4	5	6	7	8	9
$\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

We will refer to the above ten theta constants as $\theta_m(\mathbf{Z})$ with $m = 0,1,\ldots,9$ representing the ten values of $\mathbf{a}$ and $\mathbf{b}$ as defined in the above table. These are modular forms on a level 2 congruence subgroup of $Sp(g,\mathbb{Z})$ of weight 1/2. One can construct Siegel modular forms on $Sp(g,\mathbb{Z})$ using the even theta constants. For example, for $g=2$ the product of the squares of the ten even theta constants gives a cusp form of weight 10 of $Sp(2,\mathbb{Z})$ which we will encounter in the degeneracy formula for $\frac{1}{4}$-BPS states. We will look at examples of the above procedure in constructing Siegel modular forms in chapter 5.

4.9 Conclusion

In this chapter we have learnt preliminary ideas about modular forms. Modular forms will be very important to us in studying the counting of dyons in supersymmetric string theories as the degeneracy of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS states are generated by modular forms. In particular the degeneracy of $\frac{1}{4}$-BPS states are generated by genus-two Siegel modular forms. They also form the link between the CHL strings and the family of BKM Lie superalgebras related to the CHL models via the denominator identity of the BKM Lie superalgebras.

Here we have learnt the basic facts and definitions of the theory of modular forms. We have seen functions with certain restricted transformation properties under the generators of $PSL(2, \mathbb{Z})$ and how this leads to the idea of modular forms. We have also studied their Fourier expansions. We then graduated to more involved modular forms – the Siegel modular forms which are in a sense generalizations of ordinary modular forms. We studied the Fourier expansions of Siegel modular forms, besides discussing methods of constructing them. We will put these ideas to use in chapter 5 in constructing the various modular forms occuring in the counting of dyonic states.

5

Constructing the Modular Forms

5.1 Introduction

In chapter 2, we undertook an explicit counting of BPS dyonic black hole microstates in two classes of four-dimensional $\mathcal{N}=4$ supersymmetric string theories – the CHL models and the type II models. Both models are obtained as asymmetric $\mathbb{Z}_N$-orbifolds of a parent theory – the heterotic string compactified on a six-torus (for CHL models) and the type IIA string compactified on a six-torus. The degeneracy of $\frac{1}{4}$-BPS states was shown by David, Jatkar and Sen to be be generated by a genus-two Siegel modular form generalizing the proposal of DVV. Their results were restricted to prime values of N. In this chapter, we extend their proposal to all allowed values of N, not necessarily prime.

Consider a $\frac{1}{4}$-BPS dyonic state with electric charges $\mathbf{q}_e$ and $\mathbf{q}_m$. Quantization of charges imply that (for the $\mathbb{Z}_N$-orbifold)

$$\tfrac{1}{2}\mathbf{q}_e^2 = \tfrac{n}{N}\ ,\ \tfrac{1}{2}\mathbf{q}_e\cdot\mathbf{q}_m = \ell \text{ and } \tfrac{1}{2}\mathbf{q}_m^2 = m\ ,$$

for three integers (n,ℓ,m). Let $d(n,\ell,m)$ denote the degeneracy of such dyonic states. Then, the degeneracy $d(n,\ell,m)$ of dyons in the CHL models with these charges is generated by a genus-two Siegel modular form, $\widetilde{\Phi}_k(\mathbf{Z})$, at weight k and level N. One has

$$\frac{64}{\widetilde{\Phi}_k(\mathbf{Z})} = \sum_{n,\ell,m} d(n,\ell,m)\ q^{n/N} r^{\ell} s^{m}\ , \tag{5.1}$$

where factor of 64 in the numerator accounts for the degeneracy of a single $\frac{1}{4}$-multiplet. When N is prime and $N+1|24$, one has $(k+2)=24/(N+1)$ and these were the modular forms constructed by Jatkar and Sen[2]. We list below all the possible values of (N, k) that appear in the CHL models: For the type II models, one has another Siegel modular form

N	1	2	3	4	5	6	7	8	11
k	10	6	4	3	2	2	1	1	0

Table 5.1: (N, k) values for the CHL models

which we denote by $\widetilde{\Psi}_k(\mathbf{Z})$, at weight k and level N. The degeneracy of $\frac{1}{4}$-BPS dyons in the type II models are generated by

$$\frac{64}{\widetilde{\Psi}_k(\mathbf{Z})} = \sum_{n,\ell,m} d(n,\ell,m)\ q^{n/N} r^{\ell} s^{m}\ , \tag{5.2}$$

where factor of 64 in the numerator accounts for the degeneracy of a single $\frac{1}{4}$-multiplet. When N is prime and $N+1|12$, one has $(k+2) = 12/(N+1)$ and the modular forms were constructed by David, Jatkar and Sen for $N = 2, 3$[31]. We list below all the possible values of (N, k) that appear in the type II models:

N	1	2	3	4	5
k	4	2	1	1	0

(5.3)

Table 5.2: (N, k) values for the typeII models

We show that the type II modular forms $\widetilde{\Psi}_k(\mathbf{Z})$ can be written in terms ratios of the CHL modular forms $\widetilde{\Phi}_k(\mathbf{Z})$.

We also construct another closely related modular form from the two aforementioned modular forms. Let

$$\begin{aligned} \Phi_k(\mathbf{Z}) &\sim\ z_1^{-k}\ \widetilde{\Phi}_k(\widetilde{\mathbf{Z}}) \quad \text{and} \\ \Psi_k(\mathbf{Z}) &\sim\ z_1^{-k}\ \widetilde{\Psi}_k(\widetilde{\mathbf{Z}})\ , \end{aligned} \tag{5.4}$$

with

$$\tilde{z}_1 = -1/z_1 \quad , \quad \tilde{z}_2 = z_2/z_1 \quad , \quad \tilde{z}_3 = z_3 - z_2^2/z_1\ .$$

In the CHL models, the genus-two Siegel modular forms $\Phi_k(\mathbf{Z})$ are related to the R^2 corrections in the string effective action[2]. We thus have two modular forms for each class of models.

We will also need to study the 'square roots' of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$, denoted $\widetilde{\Delta}_{k/2}(\mathbf{Z})$ and $\Delta_{k/2}(\mathbf{Z})$ respectively, in order to understand the algebra structure underlying the degeneracy of the $\frac{1}{4}$-BPS states. We will obtain $\widetilde{\Delta}_{k/2}(\mathbf{Z})$ and $\Delta_{k/2}(\mathbf{Z})$, which themselves are also modular forms, in the form of an infinite sum and an infinite product along the lines of the construction of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$. These modular forms arise as the denominator formulae of BKM Lie superalgebras as we will study in the next chapter, and hence, to interpret them as the sum and product side of a denominator identity, one has to prove their modular properties which we will show in this chapter.

5.2 Modular forms via the additive lift

5.2.1 Additive lift of Jacobi forms with integer index

Now we come to the construction of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ from the weak Jacobi forms constructed as mentioned in the previous section. Given a Jacobi form of weight k and index 1, Maaß constructed a Siegel modular form of weight k leading to an explicit formula[87] using the coefficients of the Fourier expansion of the Jacobi form. This procedure is known as the *arithmetic* or *additive* lift of the Jacobi form. It is known that the ring of Siegel modular forms is generated by four modular forms with weights 4, 6, 10 and 12. For instance, the weight 10 modular form, $\Phi_{10}(\mathbf{Z})$, is generated by the Jacobi form of weight 10 and index 1

$$\phi_{10,1}(z_1, z_2) = \theta_1(z_1, z_2)^2 \; \eta(z_1)^{18} \; . \tag{5.5}$$

More generally, consider a weak Jacobi form of weight k, index 1 and level N as

$$\phi_{k,1}(z_1, z_2) = \frac{\vartheta_1(z_1, z_2)^2}{\eta(z_1)^6} \; g_\rho(z_1) = \sum_{n,\ell} a(n, \ell) \; q^n r^\ell \; , \tag{5.6}$$

where $g_\rho(z_1)$ is a genus-one modular form of weight $(k + 2)$ at level N possibly with character. We will refer to the weak Jacobi form as the *additive seed*. The Maaß construction (generalized to higher levels and modular forms with character by Jatkar and Sen[2]) leads to the following formula for weight k modular form given by the Fourier coefficients, $a(n, \ell)$, of the additive seed

$$\Phi_k(\mathbf{Z}) \equiv \sum_{(n,\ell,m)>0} \; \sum_{d|(n,\ell,m)} \chi(d) \; d^{k-1} \; a\left(\tfrac{nm}{d^2}, \tfrac{\ell}{d}\right) \; q^n r^\ell s^m \; , \tag{5.7}$$

where

$$(n, \ell, m) > 0 \text{ implies } n, m \in \mathbb{Z}_+ \ , \ \ell \in \mathbb{Z} \text{ and } (4nm - \ell^2) > 0$$

and $\chi(d)$ is a real Dirichlet character[88] modulo N. The weight k and the character χ are determined by the modular form $g_\rho(z_1)$.When $\chi(d)$ is trivial, i.e.,

$$\chi(d) = \begin{cases} 0 & \text{if } (d, N) \neq 1 \\ 1 & \text{otherwise} \ , \end{cases} \tag{5.8}$$

we obtain a level N Siegel modular form. When the Jacobi form is one with character, one sees the appearance of a non-trivial Dirichlet character and the Siegel modular form obtained from the additive lift is one with character.

5.2.2 Additive lift of Jacobi forms with half-integer index

We have just considered modular forms obtained from the additive lift of Jacobi forms with integral index. We will now study examples with half-integral index, as they appear in the denominator formulae for the the BKM Lie superalgebras $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$. that we consider in the next chapter. The simplest example of a modular form with half-integral index is given by the Jacobi theta function, $\vartheta_1(z_1, z_2)$. It is a holomorphic Jacobi form of weight 1/2 and index 1/2 with character. This Jacobi form appears as the denominator formula of the affine Kac-Moody algebra, $\widehat{A}_1^{(1)}$. Further, we will see that the modular forms $\widetilde{\Delta}_{k/2}(\mathbf{Z})$ and $\Delta_{k/2}(\mathbf{Z})$ can also be obtained as the additive lift of a Jacobi cusp form of $\Gamma_1(N)$ of weight $k/2$ and index 1/2 $\psi_{k/2,1/2}(z_1, z_2)$. The Fourier expansion of such a Jacobi form with half-integral index is of the form:

$$\psi_{k/2,1/2}(z_1, z_2) = \sum_{n,\ell \equiv 1 \bmod 2} g(n, \ell)\ q^{n/2} r^{\ell/2} \ , \tag{5.9}$$

with $q = \exp(2\pi i z_1)$ and $r = \exp(2\pi i z_2)$ and $s = \exp(2\pi i z_3)$. The modular form $\Delta_{k/2}(\mathbf{Z})$ is defined by the additive lift[7, see appendix C]:

$$\Delta_{k/2}(\mathbf{Z}) \equiv \sum_{(n,\ell,m)>0} \ \sum_{d|(n,\ell,m)} \chi(d)\ d^{\frac{k-2}{2}}\ g\left(\tfrac{nm}{d^2}, \tfrac{\ell}{d}\right)\ q^{n/2} r^{\ell/2} s^{m/2} \ , \tag{5.10}$$

where $\chi(d)$ is the character associated with the additive seed.

As an example, the Jacobi form of weight 5 and index 1/2

$$\psi_{5,1/2}(z_1,z_2)=\vartheta_1(z_1,z_2)\ \eta(z_1)^9\ , \tag{5.11}$$

generates the Siegel modular form with character, $\Delta_5(\mathbf{Z})$ via the additive lift. The Fourier expansion of the Jacobi form now involves half-integral exponents. One has

$$\psi_{5,1/2}(z_1,z_2)=\sum_{n,\ell=1 \bmod 2} g(n,\ell)\ q^{n/2}r^{\ell/2}\ , \tag{5.12}$$

with $g(n,\ell)=0$ unless $4n-\ell^2\geq 0$. The modular form $\Delta_5(\mathbf{Z})$ has the following expansion[89]

$$\Delta_5(\mathbf{Z})=\sum_{(n,\ell,m)>0}\ \sum_{d|(n,\ell,m)} d^{k-1}\ g\left(\tfrac{nm}{d^2},\tfrac{\ell}{d}\right)\ q^{n/2}r^{\ell/2}s^{m/2}\ . \tag{5.13}$$

Notice the similarity with the Maaß formula given in Eq. (5.7) with half-integral powers of q, r and s appearing where integral powers appeared. Gritsenko and Nikulin have shown that this modular form appears as the denominator formula of a BKM Lie superalgebra. $\Delta_5(\mathbf{Z})$ is a modular form with character under the full modular group, $Sp(2,\mathbb{Z})$. It transforms as

$$\Delta_5(M\cdot\mathbf{Z})=v^{\Gamma}(M)\ (C\mathbf{Z}+D)^5\ \Delta_5(\mathbf{Z})\ , \tag{5.14}$$

where $v^{\Gamma}(M)$ is the unique non-trivial real linear character of $Sp(2,\mathbb{Z})$[90] and $M=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in Sp(2,\mathbb{Z})$. An explicit expression for $v^{\Gamma}(M)$ is[89]

$$v^{\Gamma}\begin{pmatrix}0 & -I_2\\ I_2 & 0\end{pmatrix}=1\ ,\ v^{\Gamma}\begin{pmatrix}I_2 & B\\ 0 & I_2\end{pmatrix}=(-1)^{b_1+b_2+b}\ , \tag{5.15}$$

$$v^{\Gamma}\begin{pmatrix}U^T & 0\\ 0 & U^1\end{pmatrix}=(-1)^{(1+u_0+u_2)(1+u_1+u_3)+u_0u_2}\ , \tag{5.16}$$

where I_2 is the 2×2 identity matrix, $B=\left(\begin{smallmatrix}b_1 & b\\ b & b_2\end{smallmatrix}\right)$ and $U=\left(\begin{smallmatrix}u_0 & u_3\\ u_1 & u_2\end{smallmatrix}\right)$ is a uni-modular matrix.

5.3 The additive seed for CHL models

In chapter 2, we counted the states of the black hole explicitly to obtain the full partition function of the $\frac{1}{4}$-BPS states. We saw that the counting, and hence the partition function, can be split into three independent components – the degeneracy of the excitations of the Kaluza-Klein monopole, the degeneracy of the excitations of the overall motion of the D1-D5 system moving in the background of the Kaluza-Klein monopole, and the degeneracy of the relative motion of the D1-D5 system. The product of these three components gave the full partition function which was proportional to the modular form $\widetilde{\Phi}_k(\mathbf{Z})$. The product of two of the contributions, namely the degeneracy of the excitations due to the Kaluza-Klein monopole and the D1-D5 system combine to give a weak Jacobi form of weight k, index 1 and level N. This Jacobi form serves as the additive seed that generates the modular form $\widetilde{\Phi}_k(\mathbf{Z})$. This can be compared with the discussion on the Fourier-Jacobi development of a Siegel modular form at the end of chapter 4 where the corresponding Siegel modular form was broken up into a periodic piece and a Jacobi form. To generate the modular forms from the corresponding weak Jacobi form, we first need to obtain the generating function of the degeneracy of electrically charged $\frac{1}{2}$-BPS states which we denoted by $g_\rho(z_1)$ in eq. (5.6).

It turns out that the genus-one form g_ρ itself has a very nice structure which can be understood in terms of 'cycle shapes' of products of Dedekind's eta functions obtained from the set of symplectic automorphisms of a $K3$ surface[9]. The allowed cycle shapes satisfy certain conditions on the form of their exponents that depends on the orbifolding group $\mathbb{Z}_N$. This is a very interesting result that gives beautiful insight into the form of the degeneracy of the $\frac{1}{2}$-BPS states and we spend some time now understanding the degeneracy of the $\frac{1}{2}$-BPS states first before proving this result. It also provides us with the required information to construct the modular forms for the values of N not considered by Jatkar and Sen[2].

5.3.1 Counting $\frac{1}{2}$-BPS states in CHL models

The counting of the degeneracy of $\frac{1}{2}$-BPS states of a given electric charge is mapped to the counting of states of the heterotic string with the right-movers[1] in the ground state[35, 43, 42]. While this is conceptually easy to compute, for orbifolds, the contributions from the different sectors to the degeneracy need to be added up. Up to exponentially suppressed terms (for large charges), the leading contribution arises from the twisted sectors and the asymptotic expansion takes a simple form (given in Eq. (5.20) below)[35]. This asymptotic expansion is consistent with a product of η-functions called η-products. Let us, briefly recall the case of $\frac{1}{2}$-BPS states.

[1] we take the convention that right movers are taken to be supersymmetric and left movers are bosonic in the heterotic string.

Heterotic string on T^6

(Electric) $\frac{1}{2}$-BPS excitations of the heterotic string carrying charge $N \equiv \frac{1}{2}\mathbf{q}_e^2$ are obtained by choosing the supersymmetric (right-moving) sector to be in the ground state. The level matching condition becomes

$$-\frac{1}{2}\mathbf{q}_e^2 + N_L = 1 \ , \tag{5.17}$$

where $\mathbf{q}_e \in \Gamma^{22,6}$ and N_L is the oscillator contribution to L_0 in the bosonic (left-moving) sector. Thus, we see that

$$n = \tfrac{1}{2}\mathbf{q}_e^2 = N_L - 1 \ .$$

Let $d(n)$ represent the number of configurations of the heterotic string with electric charge such that $\frac{1}{2}\mathbf{q}_e^2 = n$. The level matching condition implies that we need to count the number of states with total oscillator number $N_L = (n+1)$. The generating function for this is

$$\frac{16}{\eta(z_1)^{24}} = \sum_{n=-1}^{\infty} d(n)\ q^n \ , \tag{5.18}$$

where the factor of 16 accounts for the degeneracy of a $\frac{1}{2}$-BPS multiplet – this is the degeneracy of the Ramond ground state in the right-moving sector.

The CHL orbifold of the heterotic string on T^6

In the CHL orbifold, the electric charge takes values in a lattice $\Gamma^{\perp} \subset \Gamma^{22,6}$ of signature $(22-2\hat{k},6) = (2k+2,6)$ that is not self-dual. Here $\Gamma^{\perp}$ is the sub-lattice of $\Gamma^{22,6}$ that is invariant under the action of the orbifold group. Let $\text{vol}^{\perp}$ be the volume of the unit cell in $\Gamma^{\perp}$. Define the generating function of the degeneracies $d(n)$ of $\frac{1}{2}$-BPS states as follows:

$$\frac{16}{g_\rho(z_1/N)} \equiv \sum_{n=-1}^{\infty} d(n)\ q^{n/N} \ , \tag{5.19}$$

for the $\mathbb{Z}_N$ CHL orbifold taking into account that the electric charge is quantized such that $N\mathbf{q}_e^2 \in 2\mathbb{Z}$. Setting $z_1 = i\mu/2\pi$, and in the limit $\mu \to 0$, one has[35]

$$\lim_{\mu\to 0} \frac{1}{g_\rho(i\mu/2\pi N)} = 16\ e^{4\pi^2/\mu}\ \left(\frac{\mu}{2\pi}\right)^{(k+2)/2}\ (\text{vol}^{\perp})^{1/2} + \cdots \tag{5.20}$$

where the ellipsis indicates exponentially suppressed terms. Making an ansatz for $g_\rho(z_1)$ in the form of an η-product

$$g_\rho(z_1) = \prod_{r=1}^{N} \eta(rz_1)^{a_r} = \eta(z_1)^{a_1}\eta(2z_1)^{a_2}\cdots\eta(Nz_1)^{a_N} \ , \tag{5.21}$$

we can identify the above η-product with the 'cycle shape' $\rho = 1^{a_1}2^{a_2}\cdots N^{a_N}$. The η-product has to satisfy the following conditions:

1. The asymptotic behaviour of $g_\rho(z_1)$ given in Eq. (5.20) requires

$$\begin{aligned} \big(Na_1 + N\tfrac{a_2}{2} + \cdots + a_N\big) &= 24 \ , \\ a_1 + a_2 + \cdots + a_N &= 2(k+2) \ , \\ \big(1^{a_1}2^{a_2}\cdots N^{a_N}\big)^{-1} &= \mathrm{vol}^{\perp} \ . \end{aligned} \tag{5.22}$$

   The last condition involving the volume of the unit cell is exactly what one expects for an orbifold action on the basis vectors of the self-dual lattice $\Gamma^{20,4} \subset \Gamma^{22,4}$ corresponding to the cycle shape ρ.

2. Considering $\mathbb{Z}_N$ as a cyclic permutation, one sees that the only permitted cycles are of length r such that $r|N$. One therefore imposes $a_r = 0$ unless $r|N$. Thus, when N is prime, only a_1 and a_N are non-zero and the condition simplifies considerably.

3. We will see later that the condition for a cycle to be a balanced one implies that $a_1 = a_N$ among other things. It also implies that the first equation in Eq. (5.22) can be rewritten as

$$a_1 + 2a_2 + \cdots + Na_N = 24 \ . \tag{5.23}$$

These conditions *uniquely* fix the form of $g_\rho(z_1)$. When N is prime, one sees that $a_1 = a_N = \frac{24}{N+1}$.

5.3.2 Symplectic Automorphisms of $K3$ and $\mathbf{M_{24}}$

To understand the cycle shapes that appear in the $\frac{1}{2}$-BPS state counting better, let us consider the dual description of the CHL orbifold as supersymmetric orbifold of type II string theory on $K3 \times T^2$. The orbifold group acts on the $K3$ as a symplectic (Nikulin) involution – it acts trivially on the nowhere vanishing $(2,0)$ holomorphic form. It was shown by Mukai that any finite group of symplectic automorphisms of a $K3$ surface is a subgroup of the Mathieu group, M_{23}[91].

To better understand this result, consider a symplectic automorphism of $K3$, σ, of finite order, n (it is known that $n \leq 8$). The number of fixed points, $\varepsilon(n)$ (which depends only on the order of σ) is given by

$$\varepsilon(n) = \frac{24}{n \prod_{p|n}(1+\frac{1}{p})} ,$$

and happens to match the number of fixed points for a similar element of the Mathieu group, M_{23}. The Mathieu group M_{24} can be represented as a permutation group acting on a set with 24 elements. Then, M_{23} is the subgroup of M_{24} that preserves one element of the set. Mukai the showed that if G is a finite group of symplectic automorphisms of $K3$, then

(i) G acts as a permutation on $H^*(K3, \mathbb{Z})$ and can be embedded as a subgroup of M_{23}.

(ii) G necessarily has at least five fixed points, one arising from $H^{0,0}(K3)$, $H^{2,0}(K3)$, $H^{1,1}(K3)$, $H^{0,2}(K3)$ and $H^{2,2}(K3)$. The only non-trivial part is that there is at least one fixed point in $H^{1,1}(K3)$.

The embedding of G into $M_{23} \subset M_{24}$ enables one to use known properties of M_{23}. In particular, it was shown by Conway and Norton that any element of M_{24} has a *balanced* cycle shape[92]. Recall that any permutation (of order n) may be represented by its cycle shape:

$$\rho \equiv 1^{a_1} 2^{a_2} \cdots n^{a_n} . \tag{5.24}$$

A cycle shape, ρ, is said to be balanced if there exists a positive integer M such that $\left(\frac{M}{1}\right)^{a_1}\left(\frac{M}{2}\right)^{a_2}\cdots\left(\frac{M}{n}\right)^{a_n}$ is the same as ρ. Since $\dim(H^*(K3)) = 24$, one also has the condition

$$\sum_i i\ a_i = 24 . \tag{5.25}$$

As an example, the cycle shape $1^4 2^2 4^4$ is balanced with $M = 4$ and satisfies the above condition. Now given a balanced cycle shape, ρ, consider the function $g_\rho(z_1)$ defined by the following product of η-functions:

$$\rho \longmapsto g_\rho(z_1) \equiv \eta(z_1)^{a_1}\eta(2z_1)^{a_2}\cdots\eta(nz_1)^{a_n} . \tag{5.26}$$

Note that when the condition (5.25) is satisfied, $g_\rho(z_1)$ has no fractional exponents in its

Fourier expansion about the cusp at infinity. One has

$$g_\rho(z_1) = \sum_{m=1}^{\infty} a_m\, q^m \ , \text{ with } a_1 = 1 \ , \tag{5.27}$$

where $q = \exp(2\pi i z_1)$. One more condition we require of the functions is that of *multiplicativity*. A function $g(z_1) = \sum_n a_n q^n$ is multiplicative if $a_{nm} = a_n a_m$ when $\gcd(n,m) = 1$. Of the 1575 partitions of 24 (this is equivalent to all solutions of Eq.(5.25)), Dummit et. al. have shown there exist a set of thirty multiplicative η-products each associated with a cycle that is balanced[93, 94]. Concluding the discussion on the degeneracy of $\frac{1}{2}$-BPS states, we list in Table 5.3, the various cycle shapes(restricting to shapes with $M \leq 16$), the corresponding weight of the genus-two modular form generating the degeneracies of the $\frac{1}{4}$-BPS states in the CHL model it corresponds to, and the discrete group G that is an automorphism of $K3$ which corresponds to the cycle shape ρ[9]. The groups have been identified by extracting the cycle shape from the discussion in Chaudhuri and Lowe[95] (see also proposition 5.1 in [96]). It is interesting to note that *all* cycle shapes that appear in Table 5.3 arise from the action of Nikulin involutions on $H^*(K3)$ – this includes product groups such as $\mathbb{Z}_2 \times \mathbb{Z}_2$. In examples involving product groups, the η-products are actually of level $N < M$ and the true level N is indicated in a separate column.

5.3.3 Formulae for $\Phi_k(\mathbf{Z})$ and $\widetilde{\Phi}_k(\mathbf{Z})$

We can use the expressions for $g_\rho(z_1)$ and character $\chi(d)$ from Table 5.3 to determine the additive seed and hence construct the modular form $\Phi_k(\mathbf{Z})$ for $N = 1,2,3,4,5,6,7,8,11$ using eq. (5.7). For prime N this reproduces the result of Jatkar and Sen[2].

As discussed by Jatkar and Sen[2], the generating function of dyonic degeneracies, $\widetilde{\Phi}_k(\mathbf{Z})$, is given by expansion of the modular form, $\Phi_k(\mathbf{Z})$, about another inequivalent cusp. Let

$$\widetilde{\Phi}_k(\mathbf{Z}) \equiv (\mathrm{vol}^\perp)^{1/2}\ z_1^{-k}\ \Phi_k(\widetilde{\mathbf{Z}}) \ , \tag{5.28}$$

with

$$\tilde{z}_1 = -1/z_1 \quad , \quad \tilde{z}_2 = z_2/z_1 \quad , \quad \tilde{z}_3 = z_3 - z_2^2/z_1 \ .$$

We have chosen a normalization for $\widetilde{\Phi}_k(\mathbf{Z})$ that differs from the one used in [2] but agrees with the one used in [30]. Consider $\frac{1}{4}$-BPS dyons with charges $\mathbf{q}_e$ and $\mathbf{q}_m$ such that $2n = N\mathbf{q}_e^2$, $2m = \mathbf{q}_m^2$ and $\ell = \mathbf{q}_e \cdot \mathbf{q}_m$. Then, the degeneracy $d(n,\ell,m)$ of dyons with these charges is

Cycle shape ρ	$(k+2)$	$\chi(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$	M	N	G
1^{24}	12		1	1	
$1^8 2^8$	8		2	2	$\mathbb{Z}_2$
$1^6 3^6$	6		3	3	$\mathbb{Z}_3$
2^{12}	6		4	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$1^4 2^2 4^4$	5	$\left(\frac{-1}{d}\right)$	4	4	$\mathbb{Z}_4$
$1^4 5^4$	4		5	5	$\mathbb{Z}_5$
$1^2 2^2 3^2 6^2$	4		6	6	$\mathbb{Z}_6$
$2^4 4^4$	4		8	4	$\mathbb{Z}_2 \times \mathbb{Z}_4$
3^8	4		9	3	$\mathbb{Z}_3 \times \mathbb{Z}_3$
$1^3 7^3$	3	$\left(\frac{-7}{d}\right)$	7	7	$\mathbb{Z}_7$
$1^2 2^1 4^1 8^2$	3	$\left(\frac{-2}{d}\right)$	8	8	$\mathbb{Z}_8$
$2^3 6^3$	3	$\left(\frac{-3}{d}\right)$	12	6	$\mathbb{Z}_2 \times \mathbb{Z}_6$
4^6	3	$\left(\frac{-1}{d}\right)$	16	4	$\mathbb{Z}_4 \times \mathbb{Z}_4$
$1^2 11^3$	2		11	11	$\mathbb{Z}_{11}$

Table 5.3: The function $g_\rho(z_1)$ is a modular form of weight $(k+2)$, generalized level M (true level N and character χ). Only non-trivial characters are indicated in column 3. The $N = 11$ example is not a symplectic involution of K3.

generated by

$$\frac{64}{\widetilde{\Phi}_k(\mathbf{Z})} = \sum_{n,\ell,m} d(n,\ell,m)\ q^{n/N} r^\ell s^m \ . \tag{5.29}$$

A similar additive lift for $\widetilde{\Phi}_k(\mathbf{Z})$ is given by the following seed:

$$\widetilde{\phi}_{k,1}(z_1,z_2) = \frac{\vartheta_1(z_1,z_2)^2}{\eta(z_1)^6}\ g_\rho(z_1/N)\ . \tag{5.30}$$

We now provide detailed expressions for the genus-two modular forms $\Phi_k(\mathbf{Z})$ for the CHL $\mathbb{Z}_N$ orbifolds.

$N = 1, 2, 3, 5$

For prime N and $N+1|24$, the additive seed is given by $(k+2 = 24/(N+1)$

$$\phi_{k,1}(z_1,z_2) = \vartheta_1(z_1,z_2)^2 \eta(z_1)^{k-4}\ \eta(Nz_1)^{k+2} = \sum_{n,\ell} a(n,\ell)\ q^n r^\ell\ . \tag{5.31}$$

The additive lift is ($a(n,\ell)$ is as defined by the above equation)

$$\Phi_k(\mathbf{Z}) \equiv \sum_{(n,\ell,m)>0} \ \sum_{d|(n,\ell,m)} \left(\tfrac{-1}{d}\right) d^{k-1}\ a\left(\tfrac{nm}{d^2},\tfrac{\ell}{d}\right)\ q^n r^\ell s^m\ . \tag{5.32}$$

This result is originally due to Jatkar and Sen[2]

$N = 4$

From Table 5.3, we see that $k = 3$ for $N = 4$. The seed for the additive lift is

$$\phi_{3,1}(z_1,z_2) = \frac{\vartheta_1(z_1,z_2)^2}{\eta(z_1)^2}\ \eta(2z_1)^2\eta(4z_1)^4 = \sum_{n,\ell} a(n,\ell)\ q^n r^\ell\ . \tag{5.33}$$

The additive lift is ($a(n,\ell)$ is as defined by the above equation)

$$\Phi_3(\mathbf{Z}) \equiv \sum_{(n,\ell,m)>0} \ \sum_{d|(n,\ell,m)} \left(\tfrac{-1}{d}\right) d^{k-1}\ a\left(\tfrac{nm}{d^2},\tfrac{\ell}{d}\right)\ q^n r^\ell s^m\ , \tag{5.34}$$

where the Jacobi symbol $\left(\frac{-1}{d}\right)$ is $+1$ when $d = 1 \bmod 4$; -1 when $d = 3 \bmod 4$ and 0 otherwise. This a Siegel modular form with level four and character $\psi_4(\gamma)$ where

$$\psi_4(\gamma) = \left(\frac{-1}{\det D}\right) \text{ for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_0(4)\ , \tag{5.35}$$

where $G_0(4)$ is the level four subgroup of $Sp(2,\mathbb{Z})$[97].

$N = 6$

From Table 5.3, we see that $k = 2$ for $N = 6$. The seed for the additive lift is

$$\phi_{2,1}(z_1,z_2) = \frac{\vartheta_1(z_1,z_2)^2}{\eta(z_1)^4}\ \eta(2z_1)^2\eta(3z_1)^2\eta(6z_1)^2 = \sum_{n,\ell} a(n,\ell)\ q^n r^\ell\ . \tag{5.36}$$

The additive lift is then ($a(n,\ell)$ is as defined by the above equation)

$$\Phi_2(\mathbf{Z}) \equiv \sum_{(n,\ell,m)>0} \ \sum_{\substack{d|(n,\ell,m) \\ d=1,5 \bmod 6}} d^{k-1}\ a\left(\tfrac{nm}{d^2},\tfrac{\ell}{d}\right)\ q^n r^\ell s^m\ . \tag{5.37}$$

$N = 8$

From Table 5.3, we see that $k = 1$ for $N = 8$. The seed for the additive lift is

$$\phi_{1,1}(z_1, z_2) = \frac{\vartheta_1(z_1, z_2)^2}{\eta(z_1)^4}\ \eta(2z_1)\eta(4z_1)\eta(8z_1)^2 = \sum_{n,\ell} a(n, \ell)\ q^n r^\ell\ . \tag{5.38}$$

The additive lift is then ($a(n, \ell)$ is as defined by the above equation)

$$\Phi_1(\mathbf{Z}) \equiv \sum_{(n,\ell,m)>0}\ \sum_{d|(n,\ell,m)} \left(\tfrac{-2}{d}\right) d^{k-1}\ a\left(\tfrac{nm}{d^2}, \tfrac{\ell}{d}\right)\ q^n r^\ell s^m\ , \tag{5.39}$$

where the Jacobi symbol $\left(\frac{-2}{d}\right)$ is $+1$ when $d = 1, 3 \bmod 8$; -1 when $d = 5, 7 \bmod 8$ and 0 otherwise. This is also a Siegel modular form at level eight and character $\left(\frac{-2}{\det D}\right)$.

5.3.4 Constructing the modular form $\Delta_{k/2}(\mathbf{Z})$

The square-root works only for $N = 1, 2, 3, 4, 5$. For other values of N, we find non-integral Fourier expansions arising from taking the 'square-root' of $\Phi_k(\mathbf{Z})$. In these cases, the additive seed for the modular form $\Delta_{k/2}(\mathbf{Z})$ is

$$\psi_{k/2,1/2}(z_1, z_2) = \frac{\theta_1(z_1, z_2)}{\eta(z_1)^3}\ \sqrt{g_\rho(z_1)}\ \ , \tag{5.40}$$

where $g_\rho(z_1)$ are the η-products obtained from Table 5.3. This happens to be the square root of the Jacobi form that generates $\Phi_k(\mathbf{Z})$. Similarly, the modular form $\widetilde{\Delta}_{k/2}(\mathbf{Z})$ is given by the lift of the additive seed[2] :

$$\widetilde{\psi}_{k/2,1/2}(z_1, z_2) = \frac{\theta_1(z_1, z_2)}{\eta(z_1)^3}\ \sqrt{g_\rho(z_1/N)}\ \ . \tag{5.41}$$

We have already seen the case of $N = 1$. For $N = 2, 5$, the character $\chi(d)$ is the trivial one (see Eq. (5.8)) and the corresponding modular form is got by taking the appropriate

[2]The Fourier expansion of the Jacobi form here has powers of $q^{1/N}$ Thus one has $nN \in \mathbb{Z}$ in eq. (5.10).

values of k and m. For $N = 3$, we need a non-trivial character $\chi^{\psi}(d) = \left(\frac{-3}{d}\right)$ i.e.,

$$\chi^{\psi}(d) = \begin{cases} 0 & d = 0 \bmod 3\ , \\ 1 & d = 1 \bmod 3\ , \\ -1 & d = 2 \bmod 3\ . \end{cases} \tag{5.42}$$

Thus, when $N = 3$, the weight of the modular form is even as $k = 2$. However, the seed Jacobi form, $\psi_{2,1/2}(z_1, z)$, transforms with character $w^{\gamma} w^{\psi}$ thus evading the restriction on k being odd. Taking into account the additional character, w^{ψ}, one obtains:

$$\Delta_2(\mathbf{Z}) = \sum_{(n,\ell,m)>0} \sum_{\substack{\alpha|(n,\ell,m) \\ \alpha>0}} \chi^{\psi}(\alpha)\ \alpha^{k-1}\ g\left(\tfrac{nm}{\alpha^2}, \tfrac{\ell}{\alpha}\right)\ q^{n/2} r^{\ell/2} s^{m/2}\ , \tag{5.43}$$

with $\chi^{\psi}(\alpha)$ as defined in Eq. (5.42) replacing $\chi(\alpha)$. For $N = 1, 2, 4$, we will see that $\Delta_{k/2}(\mathbf{Z})$ as well as $\widetilde{\Delta}_{k/2}(\mathbf{Z})$ can be defined as the product of k even genus-two theta constants and the additive lift is not necessary.

5.3.5 Expressions in terms of genus-two theta constants

In chapter 4 we saw that Siegel modular forms can be expressed in terms of products of even genus-two theta constants. Some of the Siegel modular forms occuring in our study also admit such an expression and we give it here. We mentioned earlier that the Siegel modular form for $N = 1$, $\Phi_{10}(\mathbf{Z})$ can be written as the squared product of all the even genus-two theta constants

$$\Phi_{10}(\mathbf{Z}) = \Big(\frac{1}{64} \prod_{m=0}^{9} \theta_m(\mathbf{Z})\Big)^2 \equiv [\Delta_5(\mathbf{Z})]^2\ . \tag{5.44}$$

Similarly, for $N = 2$ one can write the modular form $\Phi_6(\mathbf{Z})$ as products of even genus-two theta constans as follows

$$\Phi_6(\mathbf{Z}) = \Big(\frac{1}{64}\ \theta_2(\mathbf{Z}) \prod_{m=1 \bmod 2} \theta_m(\mathbf{Z})\Big)^2 \equiv [\Delta_3(\mathbf{Z})]^2\ . \tag{5.45}$$

while for $\widetilde{\Phi}_6(\mathbf{Z})$ the expression is given by

$$\widetilde{\Phi}_6(\mathbf{Z}) = \Big(\frac{1}{16}\ \theta_1(\mathbf{Z})\ \theta_3(\mathbf{Z})\ \theta_6(\mathbf{Z})\ \theta_7(\mathbf{Z})\ \theta_8(\mathbf{Z})\ \theta_9(\mathbf{Z})\ \Big)^2 \equiv \Big[\widetilde{\Delta}_3(\mathbf{Z})\Big]^2, \tag{5.46}$$

Similarly, the Siegel modular forms for the $N = 4$ example we just considered are also expressible as products of even genus-two theta functions

$$\Phi_3(\mathbf{Z}) = \left(\frac{1}{8}\ \theta_5\left(2\mathbf{Z}\right)\ \theta_7\left(2\mathbf{Z}\right)\ \theta_9\left(2\mathbf{Z}\right)\right)^2 \equiv \left[\Delta_{3/2}(\mathbf{Z})\right]^2 \ . \tag{5.47}$$

and

$$\widetilde{\Phi}_3(\mathbf{Z}) = \left(\frac{1}{4}\ \theta_8\left(\mathbf{Z}'\right)\ \theta_3\left(\mathbf{Z}'\right)\ \theta_9\left(\mathbf{Z}'\right)\right)^2 \equiv \left[\widetilde{\Delta}_{3/2}(\mathbf{Z})\right]^2 \ . \tag{5.48}$$

where $\mathbf{Z}' = \begin{pmatrix} \frac{1}{2}z_1 & z_2 \\ z_2 & 2z_3 \end{pmatrix}$.

It is pleasing to note that the formulae for $\Phi_k(\mathbf{Z})$ and $\widetilde{\Phi}_k(\mathbf{Z})$ are squares of products of even genus-two theta constants – this provides an independent way to see that their square-roots are well-defined for $N = 1, 2, 4$. We have verified these formulae by comparing the expansions from the additive lift to the one given in terms of even genus-two theta constants to a fairly high power. The representation of the modular forms in terms of the even genus-two modular forms gives us yet another way to obtain the modular forms.

5.4 Product formulae for $\Phi_k(\mathbf{Z})$ and $\widetilde{\Phi}_k(\mathbf{Z})$

Next we come to the product form of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$. The product formulae for $\Phi_k(\mathbf{Z})$ as well as $\widetilde{\Phi}_k(\mathbf{Z})$ can be given in terms of the coefficients of the Fourier expansion of the twisted elliptic genera[32]. The twisted elliptic genus for a $\mathbb{Z}_N$-orbifold of $K3$ is defined as[3]:

$$F^{m,n}(z_1, z_2) = \frac{1}{N}\mathrm{Tr}_{RR,g^m}\left((-)^{F_L+F_R} g^n q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi \imath z F_L}\right)\ , \quad 0 \leq m, n \leq (N-1)\ , \tag{5.49}$$

where g generates $\mathbb{Z}_N$ and $q = \exp(2\pi\imath z_1)$. The twisted elliptic genera are weak Jacobi forms of weight zero, index one and level N[32].

We will need to compute the $F^{m,n}(z_1, z_2)$ by use of their transformation properties under the modular group. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. Then, one has

$$F^{m,n}(z_1, z_2)\Big|_{\gamma} = F^{am+cn, bm+dn}(z_1, z_2)\ . \tag{5.50}$$

[3]The origin of these twisted elliptic genera are in threshold corrections in string theory [98, 99, 100, 101, 102]

In particular, under $T : z_1 \to z_1 + 1$ and $S : z_1 \to -1/z_1$, one has

$$F^{0,n}(z_1, z_2)\Big|_T = F^{0,n}(z_1, z_2) \quad , \quad F^{0,n}(z_1, z_2)\Big|_S = F^{n,0}(z_1, z_2) \quad . \tag{5.51}$$

More generally, the $F^{r,s}(z_1, z_2)$ are weak Jacobi forms of weight zero and index one at level N. Using their transformation properties under the modular group we can study their orbits under the action of the generators T and S of the modular group and these give constraints on the form of the $F^{m,n}(z_1, z_2)$.

Consider the Fourier expansion

$$F^{a,b}(z_1, z_2) = \sum_{m=0}^{1} \sum_{\ell \in 2\mathbb{Z}+m, n \in \mathbb{Z}/N} c_m^{a,b}(4n - \ell^2)\; q^n r^\ell \ , \tag{5.52}$$

where $q = \exp(2\pi i z_1)$ and $r = \exp(2\pi i z_2)$. We will also write $c^{a,b}(n, \ell)$ for the Fourier coefficient $c_m^{a,b}(4n - \ell^2)$. David, Jatkar and Sen provide the following product formulae using the Fourier coefficients, $c^{a,b}(4n - \ell^2)$ twisted elliptic genera[32]. One has [4]

$$\begin{aligned}\widetilde{\Phi}_k(q, r, s) =&(q^{1/N} r s) \times \prod_{m=0}^{N-1} \prod_{\substack{l,b \in \mathbb{Z}, k \in \mathbb{Z}+\frac{r}{N} \\ k,l,b>0}} \left(1 - q^k r^b s^l\right)^{\frac{1}{2}\sum_{n=0}^{N-1} e^{-2\pi i l n/N}\; c^{(m,n)}(kl,b)} \\ &\times \prod_{m=0}^{N-1} \prod_{\substack{l,b \in \mathbb{Z}, k \in \mathbb{Z}-\frac{r}{N} \\ k,l,b>0}} \left(1 - q^k r^b s^l\right)^{\frac{1}{2}\sum_{n=0}^{N-1} e^{2\pi i l n/N}\; c^{(m,n)}(kl,b)} \quad ,\end{aligned} \tag{5.53}$$

and

$$\begin{aligned}\Phi_k(q, r, s) =&(q r s) \times \prod_{m,n=0}^{N-1} \prod_{\substack{(k,l,b) \in \mathbb{Z} \\ (k,l,b)>0}} \left\{1 - e^{2\pi i m/N} q^k r^b s^l\right\}^{\frac{1}{2} c^{(m,n)}(kl,b)} \\ &\times \prod_{m,n=0}^{N-1} \prod_{\substack{(k,l,b) \in \mathbb{Z} \\ (k,l,b)>0}} \left\{1 - e^{-2\pi i m/N} q^k r^b s^l\right\}^{\frac{1}{2} c^{(m,n)}(kl,b)} \quad .\end{aligned} \tag{5.54}$$

[4]The product formula for $\widetilde{\Phi}_k$ has already been obtained from the microscopic counting considered in chapter 2.

5.4.1 Determining the twisted elliptic genera

Rather than carry out an explicit computation, we determine the twisted elliptic genera using consistency conditions based on their modular properties. When N is prime, these conditions uniquely fix the twisted elliptic genera. For composite N, there remain undetermined parameters. These parameters are fixed by imposing the condition that the product formula is compatible with the product form of the seed for the additive lift given in Eq. (5.6).We will illustrate the procedure for composite N taking the example of $\Phi_3(\mathbf{Z})$ and $\widetilde{\Phi}_3(\mathbf{Z})$, and sketching the idea of the general case from it.

Forming T-orbits

The action of T on the $F^{r,s}(z_1, z_2)$ break them up into orbits.

- We have already seen that $F^{0,s}(z_1, z_2$ are T-invariant i.e., they form orbits of length one.
- When $\gcd(r, N) = 1$, all the $F^{r,s}(z_1, z_2)$ form a single orbit of length N (under repeated action of T).
- When $\gcd(r, N) = m$, then the $F^{r,s}(z_1, z_2)$ break up into m distinct orbits of length N/m.

We will use these results to impose constraints on the form of the $F^{r,s}(z_1, z_2)$.

Along with the $F^{r,s}(z_1, z_2)$ obtained from the action of the T generator, it suffices to work out $F^{0,s}(z_1, z_2)$ and the other $F^{r,s}(z_1, z_2)$ can be obtained by the action of suitable $SL(2,\mathbb{Z})$ operations.

Let us write the most general $F^{0,s}(z_1, z_2)$. For a weak Jacobi forms of $\Gamma_0^J(N)$, $F^{0,s}(z_1, z_2)$ can be written as follows[97]:

$$\begin{aligned} F^{0,0}(z_1, z_2) &= \tfrac{8}{N} A(z_1, z_2) \quad , && (5.55)\\ F^{0,s}(z_1, z_2) &= a\ A(z_1, z_2) + \alpha_N(z_1)\ B(z_1, z_2)\ , \quad s \neq 0\ , && (5.56)\end{aligned}$$

where $\alpha_N(z_1)$ is a weight-two modular form of $\Gamma_0(N)$ and

$$A(z_1, z_2) = \sum_{i=2}^{4} \left(\frac{\vartheta_i(z_1, z_2)}{\vartheta_i(z_1, 0)} \right)^2 \quad , \quad B(z_1, z_2) = \left(\frac{\vartheta_1(z_1, z_2)}{\eta^3(z_1)} \right)^2 \ . \tag{5.57}$$

When N is composite, the dimension of modular forms at weight two is greater than one. We list the possibilities for $N = 4, 6, 8$.

$$\alpha_4(z_1) = b_1\ E_2(z_1) + b_2\ E_4(z_1)\ , \tag{5.58}$$

$$\alpha_6(z_1) = b_1\ E_2(z_1) + b_2\ E_3(z_1) + b_3\ E_6(z_1)\ , \tag{5.59}$$

$$\alpha_8(z_1) = b_1\ E_2(z_1) + b_2\ E_4(z_1) + b_3\ E_8(z_1)\ , \tag{5.60}$$

where $E_N(z_1)$ is the Eisenstein series of weight-two and level N:

$$E_N(z_1) = \tfrac{12i}{\pi(N-1)}\partial_{z_1}\left[\ln\eta(z_1) - \ln\eta(Nz_1)\right]\ ,$$

normalized so that its constant coefficient is one (See Appendix B) .

Next we use the S transformation on the ansatz for $F^{0,s}(z_1,z_2)$ and follow its transformation under powers of T and make the ansatz for $\alpha_N(z_1)$ compatible with its orbit size.

- When $(s,N) = 1$, there are no obvious constraints.
- When $(s,N) = m > 1$, then there will be constraints.
  - When $N = 4$ and $s = 2$, then $b_2 = 0$ as we need to have an orbit of size two.
  - When $N = 6$ and $s = 2, 4$, then $b_1 = b_3 = 0$ so that it is consistent with an orbit size of three.
  - When $N = 6$ and $s = 3$, then $b_2 = b_3 = 0$ so that it is consistent with an orbit size of two.
  - When $N = 8$ and $s = 2, 6$, then $b_3 = 0$ so that it is consistent with an orbit size of four.
  - When $N = 8$ and $s = 4$, then $b_2 = b_3 = 0$ so that it is consistent with an orbit size of two.

Further simplification occur from symmetry considerations. $F^{r,s}(z_1,z_2) = F^{-r,-s}(z_1,z_2)$. It implies that we have the equivalence $F^{0,s}(z_1,z_2) = F^{0,N-s}(z_1,z_2)$.

- For $N = 4$, we need to only work out $F^{0,0}(z_1,z_2)$, $F^{0,1}(z_1,z_2)$ and $F^{0,2}(z_1,z_2)$.
- For $N = 6$, we need to only work out $F^{0,0}(z_1,z_2)$, $F^{0,1}(z_1,z_2)$, $F^{0,2}(z_1,z_2)$ and $F^{0,3}(z_1,z_2)$.

- For $N = 8$, we need to only work out $F^{0,0}(z_1, z_2)$, $F^{0,1}(z_1, z_2)$, $F^{0,2}(z_1, z_2)$, $F^{0,3}(z_1, z_2)$ and $F^{0,4}(z_1, z_2)$.

We now need to fix the undetermined constants which is done by looking at the conditions on the Fourier coefficients, $c_b^{0,s}(-1)$ and $c_b^{0,s}(0)$ of $F^{0,s}(z_1, z_2)$. These two sets of numbers are related to topological objects on $K3$ and hence can be determined by studying the action of the group on $H^*(K3, \mathbb{Z})$[31]. Let $Q^{0,s}$ be the number of g^s-invariant elements of $H^*(K3, \mathbb{Z})$ (where g generates $\mathbb{Z}_N$). Also

$$Q^{0,s} = Nc_0^{0,s}(0) + 2Nc_1^{0,s}(-1) \ . \tag{5.61}$$

$Nc_1^{0,s}(-1)$ counts the number of g^s invariant $(0, 0)$ and $(0, 2)$ forms on $K3$. For symplectic involutions, these forms are invariant and hence $Nc_1^{0,s}(-1) = 2$. We thus obtain the relation

$$Nc_0^{0,s}(0) = Q^{0,s} - 4 \ . \tag{5.62}$$

Given the cycle shape one can compute the $Q^{0,s}$ as follows:

- **For prime** N: The cycle shape is $1^{k+2}N^{k+2}$. When, $s = 0$, all forms contribute and hence $Q^{0,0} = 24$. For any $s \neq 0$, one has $Q^{0,s} = k+2$. This implies that $Nc_0^{0,0}(0) = 20$ and $Nc_0^{0,s}(0) = k - 2$ for $s \neq 0$
- $N = 4$: The cycle shape is $1^4 2^2 4^4$. This implies that $Q^{0,1} = Q^{0,3} = 4$ and $Q^{0,2} = 8$. We thus obtain $4c_0^{0,s}(0) = 0$ for $s = 1, 3$ while $4c_0^{0,2}(0) = 4$.
- $N = 6$: The cycle shape is $1^2 2^2 3^2 6^2$. This implies that $Q^{0,1} = Q^{0,5} = 2$ and $Q^{0,2} = Q^{0,4} = 6$ and $Q^{0,3} = 8$. Thus one has $6c_0^{0,s}(0) = -2$ for $s = 1, 5$, $6c_0^{0,3}(0) = 4$ and $6c_0^{0,s}(0) = 2$ for $s = 2, 4$.
- $N = 8$: The cycle shape is $1^2 2^1 4^1 8^2$. This implies that $Q^{0,1} = Q^{0,3} = Q^{0,5} = Q^{0,7} = 2$ and $Q^{0,2} = Q^{0,6} = 4$ and $Q^{0,4} = 8$. Thus one has $8c_0^{0,s}(0) = -2$ for $s = 1, 3, 5, 7$, $8c_0^{0,s}(0) = 0$ for $s = 2, 6$ and $8c_0^{0,4}(0) = 4$.

Further, one has

$$c_0^{0,0}(0) = \tfrac{20}{N} \quad , \quad c_1^{0,s}(-1) = \tfrac{2}{N} \tag{5.63}$$

Also, as a consistency check on the $c_0^{0,s}(0)$ one has $k = \frac{1}{2}\sum_{s=0}^{N-1} c_0^{0,s}(0)$. For prime N all the coefficients are fixed by the above conditions and one finds the $F^{r,s}(z_1, z_2)$ for prime N are given by

$$\begin{aligned}
F^{0,0}(z_1,z_2) &= \frac{8}{N}A(z_1,z_2) \\
F^{0,s}(z_1,z_2) &= \frac{8}{N(N+1)}A(z_1,z_2) - \frac{2}{N+1}B(z_1,z_2)E_N(z_1) \\
F^{r,rk}(z_1,z_2) &= \frac{8}{N(N+1)}A(z_1,z_2) + \frac{2}{N(N+1)}B(z_1,z_2)E_N(\frac{z_1+k}{N})
\end{aligned} \tag{5.64}$$

For composite N, however, one needs more conditions to compute the $F^{r,s}(z_1,z_2)$. For $N=4$, there is one undetermined parameter in $F^{0,1}(z_1,z_2)$. For $N=6$, there are two undetermined parameters and for $N=8$, there are five undetermined parameters. These will have to be dealt with on a case by case basis. Let us choose the example of $N=4$ and illustrate the procedure for computing the $F^{r,s}(z_1,z_2)$ and from them the product form of the corresponding modular forms $\widetilde{\Phi}_3(\mathbf{Z})$ and $\Phi_3(\mathbf{Z})$.

5.4.2 Product form of $\Phi_3(\mathbf{Z})$

We start by defining

$$\widehat{F}^a(z_1,z_2) = \sum_{b=0}^{3} F^{a,b}(z_1,z_2)\ , \tag{5.65}$$

and let $\hat{c}^a(n,\ell)$ be its Fourier coefficients. The product form rewritten using the above definition as[32]

$$\Phi_3(\mathbf{Z}) = qrs\prod_{(n,\ell,m)}\Big(1-q^n r^\ell s^m\Big)^{\hat{c}^0-\hat{c}^2}\times\Big(1-\big(q^n r^\ell s^m\big)^2\Big)^{\hat{c}^2-\hat{c}^1}\times\Big(1-\big(q^n r^\ell s^m\big)^4\Big)^{\hat{c}^1} \tag{5.66}$$

where we have omitted the argument of $\hat{c}^a$ – it is (nm,ℓ) in all occurrences above to reduce the length of the equation.
Specializing the general formulae above to the case of $N=4$, we obtain

$$\begin{aligned}
\widehat{F}^0(z_1,z_2) &= \tfrac{10}{3}A(z_1,z_2) + (2b+\tfrac{1}{3})E_2(z_1)B(z_1,z_2) + (\tfrac{5}{6}-2b)E_4(z_1)B(z_1,z_2) \\
\widehat{F}^1(z_1,z_2) &= \tfrac{4}{3}A(z_1,z_2) - 2bE_2(z_1)B(z_1,z_2) - (\tfrac{5}{12}-b)E_4(z_1)B(z_1,z_2) \\
\widehat{F}^2(z_1,z_2) &= 2A(z_1,z_2) + \tfrac{1}{2}E_2(z_1)B(z_1,z_2) - (\tfrac{5}{6}-2b)E_4(z_1)B(z_1,z_2)\ ,
\end{aligned} \tag{5.67}$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are as defined in Eq. (5.57). This leads to formulae for the first two Fourier coefficients:

$$\begin{aligned} \hat{c}^0(-1) &= \tfrac{5}{6} + \tfrac{1}{3} + \tfrac{5}{6} = 2 \quad , & \hat{c}^0(0) &= \tfrac{25}{3} - \tfrac{7}{3} = 6 \\ \hat{c}^1(-1) &= \tfrac{1}{3} - \tfrac{5}{12} - b = -b - \tfrac{1}{12} \quad , & \hat{c}^1(0) &= \tfrac{25}{6} + 2b \\ \hat{c}^2(-1) &= \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{5}{6} + 2b = 2b + \tfrac{1}{6} \quad , & \hat{c}^2(0) &= \tfrac{17}{3} - 4b \end{aligned} \tag{5.68}$$

We need $\hat{c}^1(-1) = \hat{c}^2(-1) = 0$ else we will have terms of the type $(1 - r^2)$ and $(1 - r^4)$ in the product expansion for $\Phi_3(\mathbf{Z})$. This fixes the unfixed constant $b = -1/12$. We can now write out all the terms with $m = 0$ in the product formulae as we now have determined that $\hat{c}^1(0) = 4$ and $\hat{c}^2(0) = 6$. These give rise to terms of the form

$$\prod_{n=1}^{\infty} (1 - q^n)^0 (1 - q^{2n})^2 (1 - q^{4n})^4 \ .$$

This agrees with the (infinite set of) terms that appear from the product expansion of the additive seed:

$$\phi_{3,1}(z_1, z_2) = \frac{\vartheta_1^2(z_1, z_2)}{\eta(z_1)^6} \eta(z_1)^4 \eta(2z_1)^2 \eta(4z_1)^4 \ .$$

Since we have fixed the constant b, we can now write exact expressions for the $F^{a,b}(z_1, z_2)$.

$$\begin{aligned} F^{0,0}(z_1,z_2) &= 2A(z_1,z_2) \\ F^{0,1}(z_1,z_2) &= F^{0,3}(z_1,z_2) = \tfrac{1}{3}A(z_1,z_2) + \Big[-\tfrac{1}{12}E_2(z_1) + \tfrac{1}{2}E_4(z_1) \Big] B(z_1,z_2) \\ F^{0,2}(z_1,z_2) &= \tfrac{2}{3}A(z_1,z_2) + \tfrac{1}{3}E_2(z_1)B(z_1,z_2) \\ F^{1,k}(z_1,z_2) &= F^{3,3k}(z_1,z_2) = \tfrac{1}{3}A(z_1,z_2) + \Big[-\tfrac{1}{24}E_2\big(\tfrac{z_1+k}{2}\big) + \tfrac{1}{8}E_4\big(\tfrac{z_1+k}{4}\big) \Big] B(z_1,z_2) \\ F^{2,2k}(z_1,z_2) &= \tfrac{2}{3}A(z_1,z_2) - \tfrac{1}{6}E_2\big(\tfrac{z_1+k}{2}\big)B(z_1,z_2) \\ F^{2,2k+1}(z_1,z_2) &= \tfrac{1}{3}A(z_1,z_2) + \Big[\tfrac{5}{12}E_2(z_1) - \tfrac{1}{2}E_4(z_1) \Big] B(z_1,z_2) \end{aligned} \tag{5.69}$$

and

$$\begin{aligned} \widehat{F}^0(z_1,z_2) &= \tfrac{10}{3}A(z_1,z_2) + \tfrac{1}{6}E_2(z_1)B(z_1,z_2) + E_4(z_1)B(z_1,z_2) \\ \widehat{F}^1(z_1,z_2) &= \tfrac{4}{3}A(z_1,z_2) + \tfrac{1}{6}E_2(z_1)B(z_1,z_2) - \tfrac{1}{2}E_4(z_1)B(z_1,z_2) \\ \widehat{F}^2(z_1,z_2) &= 2A(z_1,z_2) + \tfrac{1}{2}E_2(z_1)B(z_1,z_2) - E_4(z_1)B(z_1,z_2) \end{aligned} \tag{5.70}$$

5.4.3 Product Formula for $\widetilde{\Phi}_3(\mathbf{Z})$

The product formula for $\widetilde{\Phi}_3(\mathbf{Z})$ is

$$\widetilde{\Phi}_3(\mathbf{Z}) = q^{1/4} rs \prod_{a}^{3} \prod_{\substack{\ell,m\in\mathbb{Z},\\ n\in\mathbb{Z}+\frac{a}{4}}} \Big(1 - q^n r^\ell s^m\Big)^{\sum_{b=0}^{3} \omega^{-bm} c^{(a,b)}(4nm-\ell^2)} \tag{5.71}$$

where $\omega = \exp(\frac{2\pi\imath}{3})$ is a cube root of unity, $c^{(a,b)}(4nm-\ell^2)$ are the Fourier coefficients of the twisted elliptic genera, $F^{(a,b)}(z_1, z_2)$. One can also prove that all the exponents that appear in the product formulae for $\Phi_3(\mathbf{Z})$ and $\widetilde{\Phi}_3(\mathbf{Z})$ are all even integers. One can show that the following expressions

$$[4A(z_1,z_2) - B(z_1,z_2)]\,/12\ , \quad [E_2(z_1) - 1]/24 \ \text{ and } [E_4(z_1) - 1]/8$$

all have integral Fourier coefficients[8, see appendix A]. A straightforward but tedious computation then shows that all exponents are even integers. This will be important to us when we construct the product forms of the modular forms $\widetilde{\Delta}_{k/2}(\mathbf{Z})$ and $\Delta_{k/2}(\mathbf{Z})$ as 'square roots' of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ and need the exponents to be even integers for the operation of taking square roots to be valid.

On the sum side, the integrality of coefficients in the Fourier expansion follows from the integrality properties of the genus-two theta constants.

5.5 The additive seed for type II models

We will now construct the modular forms, $\Psi_k(\mathbf{Z})$ and $\Psi_k(\mathbf{Z})$, for the type II models via the additive lift. The basic idea is similar to what was done for the CHL models. We first obtain the generating function for electrically charged $\frac{1}{2}$-BPS states – call it $g_\rho(z_1)$ as before. Then the additive seed is as in the CHL models (eq. (5.6)). We will see that the multiplicative η-products that appeared in the CHL model get replaced by η-quotients. This reflects the fact that electrically charged states in the type II model arise from bosonic left-movers of the type IIA string (See also the discussion in 2.6.1).

5.5.1 Counting electrically charged $\frac{1}{2}$-BPS states

As mentioned earlier, we will define our charge in the second description. In this case, electrically charged states appear as excitations of the type IIA string. In particular, the degeneracy is dominated by the contribution from the twisted sector states. We will compute the electrically charged states in a twisted sector. $\frac{1}{2}$-BPS states arise when the right-movers are in the ground state and we allow all excitations that are consistent with level matching.

$N = 1$

As a warm-up, consider the left-movers of the type IIA string on T^6. In the Ramond sector and in the light-cone gauge, one has eight periodic bosons and periodic fermions. All oscillators, bosonic and fermionic, have integer moding and the Witten index is given by the product of the bosonic (indicated by $\mathcal{W}_B$)and fermionic contributions (indicated by $\mathcal{W}_F$):

$$\mathcal{W}_B \times \mathcal{W}_F = \left(\frac{1}{\prod_n (1-q^n)}\right)^8 \times \left(\prod_n (1-q^n)\right)^8 = 1 \ . \tag{5.72}$$

Note that we have not considered the zero-modes. This is expected as there is a perfect cancellation of bosonic and fermionic contributions in the Witten index. Of course, the oscillator partition function is not unity and equals

$$\mathcal{Z}_B \times \mathcal{Z}_F = \left(\frac{1}{\prod_n (1-q^n)}\right)^8 \times \left(\prod_n (1+q^n)\right)^8 = \frac{\eta(2\tau)^8}{\eta(\tau)^{16}} \ . \tag{5.73}$$

Interestingly, this is quotient of η-functions at level 2 (This appears in the construction of the fake Monster Lie superalgebra [79])

$N = 2$

The eight periodic bosons have integer moding and each contribute a factor of $\eta(\tau)^{-1}$ to the Witten index while the eight anti-periodic fermions each have half-integer moding and contribute $\eta(\tau/2)/\eta(\tau)$. One has

$$\begin{aligned}\mathcal{W}_B \times \mathcal{W}_F &= \left(\frac{1}{\prod_n(1-q^n)}\right)^8 \times \left(\prod_n (1-q^{n+1/2})\right)^8 \\ &= \frac{\eta(\tau/2)^8}{\eta(\tau)^{16}} = \frac{1}{g_{\tilde{\rho}}(\tau/2)} \ ,\end{aligned} \tag{5.74}$$

where the frame shape $\tilde{\rho} = 1^{-8}2^{16}$.

Recall that cycle shapes represent conjugacy classes of a permutation. Frame shapes generalize this notion to conjugacy classes of elements of arbitrary discrete groups. In our example, the discrete group turns out to be the Conway group Co_1 [103] as we discuss later.

$N = 3$

The six periodic bosons have integer moding and each contribute a factor of $\eta(\tau)^{-1}$ to the Witten index. While the two other bosons have moding fractional moding of $\pm 1/3$. The fermions each have fractional moding of $\pm 1/3$ and contribute $\eta(\tau/2)/\eta(\tau)$. One has

$$\begin{aligned}\mathcal{W}_B \times \mathcal{W}_F &= \frac{1}{\prod_n(1-q^n)^6(1-q^{n+1/3})(1-q^{n-1/3})} \times \prod_n (1-q^{n+1/3})^4(1-q^{n-1/3})^4 \\ &= \frac{\eta(\tau/3)^3}{\eta(\tau)^9} = \frac{1}{g_{\tilde{\rho}}(\tau/3)} \ ,\end{aligned} \tag{5.75}$$

where the frame shape $\tilde{\rho} = 1^{-3}3^{9}$.

$N = 4$

The six periodic bosons have integer moding and each contribute a factor of $\eta(\tau)^{-1}$ to the Witten index. While the two other bosons are antiperiodic and have moding fractional half-integral moding. The fermions each have fractional moding of $\pm 1/4$ and contribute $\eta(\tau/2)/\eta(\tau)$. One has

$$\begin{aligned}\mathcal{W}_B \times \mathcal{W}_F &= \frac{1}{\prod_n(1-q^n)^6(1-q^{n+1/2})^2} \times \prod_n (1-q^{n+1/4})^4(1-q^{n-1/4})^4 \\ &= \frac{\eta(\tau/4)^4}{\eta(\tau)^4\eta(\tau/2)^6} = \frac{1}{g_{\tilde{\rho}}(\tau/4)} \ ,\end{aligned} \tag{5.76}$$

where the frame shape $\tilde{\rho} = 1^{-4}2^{6}4^{4}$.

$N = 5$

Four bosons have integer moding while the other four have fractional moding of $r/5$ with $r = 1, 2, 3, 4$. The fermions appear with fractional moding of $r/5$ with $r = 1, 2, 3, 4$ occurring in pairs. One has

$$\begin{aligned} \mathcal{W}_B \times \mathcal{W}_F &= \frac{1}{\prod_n (1-q^n)^4 \prod_{r=1}^4 (1-q^{n+r/5})} \times (\prod_n \prod_{r=1}^4 (1-q^{n+r/5}))^2 \\ &= \frac{\eta(\tau/5)}{\eta(\tau)^5} = \frac{1}{g_{\tilde{\rho}}(\tau/5)} \ , \end{aligned} \tag{5.77}$$

where the frame shape $\tilde{\rho} = 1^{-1}5^5$.

Multiplicative η-quotients

The counting of $\frac{1}{2}$-BPS states is given by η-quotients that are associated with the frame shapes $\tilde{\rho}$ given in Table 5.4. This nicely generalizes the corresponding result for CHL strings where the generating functions were given by η-products corresponding to cycle shapes.

The appearance of the η-quotients and frame shapes may be understood as follows. It is known that the Conway group Co_1 arise as the group of automorphisms of algebra of chiral vertex operators in the NS sector of superstring[104]. Any symmetry of finite order of the chiral superstring *must* thus be an element of Co_1. It is known that the conjugacy classes of Co_1 are given by frame shapes.

Multiplicative η-quotients have been studied by Martin[105] and he has provided a list of 71 such quotients – almost all appear to be associated to conjugacy classes. Table 5.4 is a subset of this list excluding the ones $N = 2$. The η-quotients for $N = 2$ violate the multiplicative condition of Martin – he requires them to be eigenforms of all Hecke operators. The one's for $N = 2$ are not eigenforms for T_2 as can be easily checked[5] It appears possible that the condition imposed by Martin might be too strong and hence we may need to look for a weaker condition.

The η-quotients for $N = 2, 3$ have been derived in [31] and our results agree with the expressions given there.

[5]We thank Martin for useful correspondence which clarified this point.

k	$\widetilde{\rho}$	ρ	$\chi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$	N	G
2	$1^{-8}2^{16}$	$1^{16}2^{-8}$		2	$\mathbb{Z}_2$
1	$1^{-3}3^{9}$	$1^{9}3^{-3}$	$\left(\frac{-3}{d}\right)$	3	$\mathbb{Z}_3$
1	$1^{-4}2^{6}4^{4}$	$1^{4}2^{6}4^{-4}$	$\left(\frac{-1}{d}\right)$	4	$\mathbb{Z}_4$
0	$1^{-1}5^{5}$	$1^{5}5^{-1}$		5	$\mathbb{Z}_5$

Table 5.4: η-quotients with $N \leq 5$: ρ is the frame shape, $k+2$ is the weight of the η-quotient.

5.5.2 Product Formulae for the type II models

David, Jatkar and Sen have provided product formulae for the $N = 2, 3$ type II models[31]. As for the CHL models, there are given in terms of the twisted elliptic genus for T^4. The product formulae for $\Psi_k(\mathbf{Z})$ and $\widetilde{\Psi}_k(\mathbf{Z})$ are identical to those appearing in the CHL models – eq. (5.53) and (5.54) – the coefficients used are however those from the type II twisted elliptic genus. For $N = 2, 3$, $F^{(r,s)}(\tau, z)$, David, Jatkar and Sen find

$$\begin{aligned}
F^{(0,0)}(\tau,z) &= 0 \\
F^{(0,s)}(\tau,z) &= \frac{16}{N}\sin^4\left(\frac{\pi s}{N}\right)\frac{\vartheta_1\left(\tau, z+\frac{s}{N}\right)\vartheta_1\left(\tau,-z+\frac{s}{N}\right)}{\vartheta_1\left(\frac{s}{N}\right)^2} \\
&\qquad \text{for } 1 \leq s \leq N-1\,, \\
F^{(r,s)}(\tau,z) &= \frac{4\,N}{(N-1)^2}\,\frac{\vartheta_1\left(\tau, z+\frac{s}{N}+\frac{r}{N}\tau\right)\vartheta_1\left(\tau,-z+\frac{s}{N}+\frac{r}{N}\tau\right)}{\vartheta_1\left(\frac{s}{N}+\frac{r}{N}\tau\right)^2}\,, \\
&\qquad \text{for } 1 \leq r \leq N-1,\ 0 \leq s \leq N-1\,.
\end{aligned} \tag{5.78}$$

The twisted elliptic genera for type II models can be rewritten in terms of the elliptic genera that appear in the CHL models. For the $\mathbb{Z}_2$ orbifold of the type II model, $F^{(r,s)}(\tau, z)$ can be written as

$$F^{(r,s)}_{II}(\tau,z) = 2F^{(r,s)}_{N=2\ CHL}(\tau,z) - F^{(r,s)}_{N=1\ Het.}(\tau,z), \tag{5.79}$$

and for the $\mathbb{Z}_3$ orbifold of the type II model, they can be written as

$$F^{(r,s)}_{II}(\tau,z) = \frac{3}{2}F^{(r,s)}_{N=3\ CHL}(\tau,z) - \frac{1}{2}F^{(r,s)}_{N=1\ Het.}(\tau,z). \tag{5.80}$$

This implies that the type II modular forms for $N = 2, 3$ can be written in terms of the Siegel modular forms for the CHL models. In order to see this, we rewrite the η-quotients that appear for $N = 2$ in a suggestive manner as follows

$$\begin{aligned} g_4(\tau) &= \frac{\eta^{16}(2\tau)}{\eta^8(\tau)} = \frac{\eta^{16}(2\tau)\eta^{16}(\tau)}{\eta^{24}(\tau)}, \\ g_3(\tau) &= \frac{\eta^9(3\tau)}{\eta^3(\tau)} = \frac{\eta^9(3\tau)\eta^9(\tau)}{\eta^{12}(\tau)}. \end{aligned} \tag{5.81}$$

In this form it is evident that the modular form g_4 is a ratio of two modular forms, with the numerator corresponding to the square of the cusp form which counts half BPS states in the $\mathbb{Z}_2$ CHL model and the denominator is a cusp form which counts half BPS states in the heterotic string theory. This naturally suggest that the Siegel modular form for type II $\mathbb{Z}_2$ model is a ratio of Siegel modular forms,

$$\Psi_2(\mathbf{Z}) = \frac{\Phi_6(\mathbf{Z})^2}{\Phi_{10}(\mathbf{Z})} . \tag{5.82}$$

In the $\mathbb{Z}_3$ case, we find that g_3 is again a ratio, suggesting the relation

$$\Psi_1(\mathbf{Z}) = \frac{\Delta_2(\mathbf{Z})^3}{\Delta_5(\mathbf{Z})}, \tag{5.83}$$

One can easily see that both these identities follow from the product formulae using the relation between type II and CHL twisted elliptic genera given in eq. (5.79) and (5.80). Further, it also follows that a similar relationship holds for the other modular forms.

$$\begin{aligned} \widetilde{\Psi}_2(\mathbf{Z}) &= \frac{\widetilde{\Phi}_6(\mathbf{Z})^2}{\Phi_{10}(\mathbf{Z})} \\ \widetilde{\Psi}_1(\mathbf{Z}) &= \frac{\widetilde{\Delta}_2(\mathbf{Z})^3}{\Delta_5(\mathbf{Z})} \end{aligned} \tag{5.84}$$

We conclude this section with conjectural formulae for the $N = 4$ type II model:

$$\widetilde{\Psi}_1(\mathbf{Z}) = \frac{\Delta_3(\mathbf{Z})\Delta_{3/2}(\mathbf{Z})^2}{\Delta_5(\mathbf{Z})} \quad \text{and} \quad \Psi_1(\mathbf{Z}) = \frac{\widetilde{\Delta}_3(\mathbf{Z})\widetilde{\Delta}_{3/2}(\mathbf{Z})^2}{\Delta_5(\mathbf{Z})} . \tag{5.85}$$

5.6 Conclusion

In this chapter we have studied the various modular forms that appear in the couting of dyonic states in $\mathcal{N} = 4$ string theories that we are studying in this thesis. The degeneracy of the electrically charged $\frac{1}{4}$-BPS states are generated by a product of η-functions that are associated with cycle shapes. Their generalization, given by η-quotients that are associated to the frame shapes give the generating function of the degneracies of the $\frac{1}{4}$-BPS states in the type II models. The η-products give a nice way of relating the degeneracy of the $\frac{1}{4}$-BPS states to the symplectic automorphisms of the $K3$ surface. Similarly, the η-quotients are related to the conjugacy classes of Co_1, which are given by frame shapes.

The degeneracy of the $\frac{1}{4}$-BPS states are given by genus-two Siegel modular forms $\widetilde{\Phi}_k(\mathbf{Z})$. Also, the string R^2 corrections are given by another modular form, denoted $\Phi_k(\mathbf{Z})$. In this chapter we have studied the construction of these modular forms in more than one ways. The additive lift gives the modular forms as an infinite sum. The construction of the genus-two Siegel modular forms from an additive lift was discussed as the Fourier-Jacobi development of Siegel modular forms in chapter 4. The modular forms are constructed from a seed which is a weak Jacobi form of the same weight and index 1 and level N. The weak Jacobi form is obtained from the cycle shape ρ is $\frac{\vartheta_1(z_1,z_2)^2}{\eta(z_1)^6}\ g_rho(z_1)$. The modular forms were also constructed as an infinite product with exponents related to the twisted elliptic genera of $K3$. We have also seen expressions for the modular forms as products of even genus-two theta constants in some cases.

The same procedure was used to obtain the modular forms $\widetilde{\Delta}_k(\mathbf{Z})$ and $\Delta_k(\mathbf{Z})$ which are the 'square roots' of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ respectively. They were constructed as the additive lifts of weak Jacobi forms with half-integer indices. The fact that all the exponents of the product form of $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ are even integers immediately yields the product form of $\widetilde{\Delta}_k(\mathbf{Z})$ and $\Delta_k(\mathbf{Z})$. In addition, some of the modular forms have been obtained as products of even genus-two theta constants.

It is important to obtain the modular forms in the sum and product forms separately. This is useful when we relate them to the denominator identity of BKM Lie superalgebras. We will study this idea in the next chapter where we understand the relation between the CHL strings and the family of BKM Lie superalgebras that are related to them with the modular forms being the bridge between the two.

6

BKM Lie superalgebras From Dyon Spectra

6.1 Introduction

In this chapter we focus on the algebraic side of the degeneracy of $\frac{1}{4}$-BPS states. As mentioned previously, there is an algebraic structure underlying the degeneracy of the $\frac{1}{4}$-BPS states, given by a family of BKM Lie superalgebras. These BKM Lie superalgebra are related to the dyonic degeneracies via the modular forms generating the degeneracies of the $\frac{1}{4}$-BPS states and R^2 corrections to the string action. These modular forms occur as the denominator formulae of the various BKM Lie superalgebras. We will explore this idea in this chapter, studying the BKM Lie superalgebras corresponding to the various CHL models. The discovery of new BKM Lie superalgebras has been one of the main results of the work presented in this thesis.

6.2 The Algebra of $\frac{1}{4}$-BPS States

There has emerged a promising new direction by studying the algebra satisfied by the degeneracy of the $\frac{1}{4}$-BPS states. The 'square roots' of the genus-two modular forms generating the degeneracies have been found to be related to a general class of infinite-dimensional Lie algebras known as Borcherds-Kac-Moody (BKM) Lie superalgebras and this endows the degeneracy of the $\frac{1}{4}$-BPS states with an underlying BKM Lie superalgebra structure[17, 7, 8]. Following this insight, physical ideas of the theory such as the structure of the walls of marginal stability[33] have been understood from an algebraic point of view as the walls of the fundamental Weyl chamber[17, 8].

It was observed by DVV that the Siegel modular form constructed by them, that generated the degeneracy of $\frac{1}{4}$-BPS states, was also studied by Gritsenko and Nikulin in the context of an infinite-dimensional BKM Lie superalgebra[86]. More precisely, the modular form $\Phi_{10}(\mathbf{Z})$ is the denominator identity of the BKM Lie superalgebra that Gritsenko and Nikulin studied. When Sen and Jatkar constructed the modular forms that generate the degeneracy of $\frac{1}{4}$-BPS states in CHL models, it was natural to look for an underlying algebraic structure along the lines of the $N = 1$ models. This was studied in [17, 7, 8, 9]. We summarize the results below.

6.3 The BKM Lie superalgebra $\mathcal{G}_1$

The BKM Lie superalgebra corresponding to the CHL model without any orbifolding with its denominator identity given by the square root of the modular form $\Phi_{10}(\mathbf{Z})$ was studied by Gritsenko and Nikulin[86]. We denote it by $\mathcal{G}_1$, where the subscript denotes the N of the orbifolding group $\mathbb{Z}_N$. The Cartan matrix of $\mathcal{G}_1$ is given by

$$A_{1,II} \equiv \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} . \tag{6.1}$$

It is a rank 3 hyperbolic matrix, as one of the eigenvlaues of the Cartan matrix is negative. The algebra $\mathcal{G}_1$ has three real simple roots, call them δ_1, δ_2 and δ_3 whose Gram matrix (matrix of inner products) is $A_{1,II}$. The three real simple roots define the root lattice $M_{II} = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3$ and a fundamental polyhedron, $\mathcal{M}_{II}$, which is given by the region bounded by the spaces orthogonal to the real simple roots.

$$\mathbb{R}_+\mathcal{M}_{II} = \{x \in M_{II} \otimes \mathbb{R} \mid (x, \delta_i) \leq 0, i = 1, 2, 3\}. \tag{6.2}$$

Let us write the roots in terms of a basis (f_2, f_3, f_{-2}) which are related to the δ_i in the following way:

$$\delta_1 = 2f_2 - f_3, \quad \delta_2 = f_3, \quad \delta_3 = 2f_{-2} - f_3 . \tag{6.3}$$

The non-vanishing inner products among the elements f_i are:

$$(f_2, f_{-2}) = -1, \quad (f_3, f_3) = 2 . \tag{6.4}$$

Thus, (f_2, f_3, f_{-2}) provide a basis for Minkowski space $\mathbb{R}^{2,1}$. Consider the time-like region

$$V = \{x \in \mathbb{R}^{2,1} \mid (x,x) < 0\} ,$$

in $\mathbb{R}^{2,1}$. Let V^+ denote the future light-cone in the space and

$$\mathbf{Z} = z_3 f_2 + z_2 f_3 + z_1 f_{-2} , \tag{6.5}$$

be such that $\mathbf{Z} \in \mathbb{R}^{2,1} + iV^+$. This is equivalent to $\mathbf{Z} \in \mathbb{H}_2$, the Siegel upper-half space[86]. In addition to the three real simple roots, there are three primitive light-like vectors, i.e. $(\eta,\eta) = 0 :\ 2f_2,\ 2f_{-2}$ and $(2f_{-2} - 2f_3 + 2f_2)$ each with multiplicity 9 and two primitive vectors satisfying $(\eta,\eta) < 0 :\ (2f_{-2} + 2f_2)$ and $(2f_{-2} - f_3 + 2f_2)$. These roots are imaginary since their norm is not positive definite, i.e. $(\eta,\eta) \leq 0$. The imaginary light-like roots are generated by the formula

$$1 - \sum_{t\in\mathbb{N}} m(t\eta_0) q^t = \prod_{k\in\mathbb{N}} (1-q^k)^9 = \frac{\sqrt{f^{(10)}(\tau)}}{\eta(\tau)^3} . \tag{6.6}$$

Negative value of multiplicity implies that the root is fermionic. For instance, one has $m(2\eta_0) = -27$. Thus, such roots are fermionic and hence we have a superalgebra. The imaginary simple roots belong to the space $M_{II} \cap \mathbb{R}_+\mathcal{M}_{II}$. Let us look at the Weyl group of the BKM Lie superalgebra $\mathcal{G}_1$.

6.3.1 The Weyl Group $\mathcal{W}(A_{1,II})$

Given the three real simple roots $(\delta_1, \delta_2, \delta_3)$, whose Gram matrixis given by the matrix $A_{1,II}$, the Weyl group, $\mathcal{W}(A_{1,II})$, is the group generated by the three elementary reflections, (w_1, w_2, w_3), with respect to the three real simple roots. The Weyl group $\mathcal{W}(A_{1,II})$ can be written as a normal subgroup of $PGL(2,\mathbb{Z})$. Recall that $PGL(2,\mathbb{Z})$ is given by the integral matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = \pm 1$. One has[86] (see also [106, 17])

$$PGL(2,\mathbb{Z}) = \mathcal{W}(A_{1,II}) \rtimes S_3 , \tag{6.7}$$

where S_3 is the group of permutations of the three real simple roots. Also, the lattice M_{II} has a *lattice Weyl vector* which is an element $\rho \in M_{II} \otimes \mathbb{Q}$ such that all the real simple roots satisfy[1]

$$(\rho, \delta_i) = -\frac{(\delta_i, \delta_i)}{2} = -1 \ . \tag{6.8}$$

One has $\rho = (\delta_1 + \delta_2 + \delta_3)/2$ i.e., it is one-half of the sum over real simple roots. The positive real roots are then given by

$$L_+^{\text{re}} = \Big(\mathcal{W}(\delta_1, \delta_2, \delta_3) \cap M_{II}^+\Big) \ , \tag{6.9}$$

where $\mathcal{W}$ refers to the Weyl group $\mathcal{W}(A_{1,II})$ and $M_{II}^+ = \mathbb{Z}_+\delta_1 \oplus \mathbb{Z}_+\delta_2 \oplus \mathbb{Z}_+\delta_3$.

Choosing a set of matrices for the basis f_2, f_3, f_{-2}, we can make the action of the Weyl group explicit. Consider the following identification:

$$f_{-2} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ f_3 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ f_2 \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{6.10}$$

With the above identification, the root vectors are given by the matrices

$$\delta_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \delta_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ , \ \delta_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \ . \tag{6.11}$$

One also has $\rho^{(4)} = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ in agreement with the general formula given in ref. [8, see Eq. 5.2]. In terms of the variables q, r, s these real simple roots are r^{-1}, qr and sr respectively. The norm of a matrix $N \in M_{1,0}$ is then given by $-2\det N$. The Weyl group has the following action:

$$N \to A \cdot N \cdot A^T \ , \quad A \in PGL(2, \mathbb{Z}) \text{ and } N \in M_{1,0} \ . \tag{6.12}$$

The S_3 mentioned in Eq. (6.7) is generated by

$$r_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad r_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \ . \tag{6.13}$$

satisfying $r_{-1}^2 = r_0^2 = (r_{-1}r_0)^3 = 1$. The three elementary reflections that generate $\mathcal{W}(A_{1,II})$

[1]The standard convention is to define ρ through the condition $(\rho, \delta_i) = (\delta_i, \delta_i)/2$ for all real simple roots δ_i. However, we reproduce the notation of Gritsenko and Nikulin [86] (which differs by a sign) here.

are given by the following $PGL(2,\mathbb{Z})$ matrices:

$$w_{\delta_1} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \ , \ w_{\delta_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ w_{\delta_3} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \ . \tag{6.14}$$

6.3.2 The Weyl Chambers

$\mathcal{G}_1$ is an elliptic BKM Lie superalgebra. This means that the volume of the fundamental Weyl chamber is finite. This is an important property and ensures many nice properties for the BKM Lie superalgebra. Gritsenko and Nikulin have classified rank-three hyperbolic BKM Lie superalgebra admitting a lattice Weyl vector with finite volume of the fundamental Weyl chamber[107](see also [108, 109]). Recall from the discussion about the Weyl chambers of Lie algebras in Chapter 3 that the choice of the basis of simple roots determine the fundamental Weyl chamber. A different, but equivalent, basis of simple roots will give a different Weyl chamber which is related to the first Weyl chamber through Weyl reflections. The Weyl group acts simply and transitively on the set of Weyl chambers. Cheng and Verlinde have studied the walls of the Weyl chamber in relation to the moduli space of the CHL string and found that the walls of the Weyl chamber of the BKM Lie superalgebra can be identified with domains in the moduli space, specifically, they coincide with the walls of marginal stability of the $\frac{1}{4}$-BPS states of the theory [17]. We will now summarize their arguments and give a correspondence between the walls of the Weyl chambers of the BKM Lie superalgebras and the walls of marginal stability.

Cheng and Verlinde[17] and Cheng and Dabholkar[8] have shown the for $N = 1, 2, 3$ CHL models, the fundamental domains are the Weyl chambers of a family of rank-three BKM Lie superalgebras. This was extended to the $N = 4$ case in [9]. Each wall (edge) of the fundamental domain is identified with a real simple root of the BKM Lie superalgebra. Recall that we saw in chapter 2 that each wall corresponds to a pair of rational numbers $(\frac{b}{a}, \frac{d}{c})$. This is related to a real simple root α of the BKM Lie superalgebra as follows:

$$(\tfrac{b}{a}, \tfrac{d}{c}) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \alpha = \begin{pmatrix} 2bd & ad+bc \\ ad+bc & 2ac \end{pmatrix} \ , \tag{6.15}$$

with $ac \in N\mathbb{Z}$ and $ad, bc, bd \in \mathbb{Z}$. The norm of the root is[17]

$$-2\det(\alpha) = 2(ad-bc)^2 = 2 \ .$$

The Cartan matrix, $A^{(N)}$, is generated by the matrix of inner products among all real simple roots. For instance, $A^{(1)} = A_{1,II}$ defined in Eq. (7.2).

The 'square root' of the modular form $\widetilde{\Phi}_k(\mathbf{Z})$ that generates dyon degeneracies, $\widetilde{\Delta}_{k/2}(\mathbf{Z})$, is related to the Weyl-Kac-Borcherds denominator formula via its additive and multiplicative lifts. Finally, the extended S-duality group is given by[2]

$$\mathcal{W}(A^{(N)}) \rtimes D_N \ , \tag{6.16}$$

where $\mathcal{W}(A^{(N)})$ is the Weyl group generated by Weyl reflections of all the simple real roots[3] and D_N is the dihedral group that is the symmetry group of the polygon corresponding to the Weyl chamber.

6.3.3 The Denominator Formula

Now we come to the most important part of the connection between the CHL strings and the BKM Lie superalgebras. The connection to the CHL strings of the BKM Lie superalgebra comes from the denominator formula. Gritsenko and Nikulin have shown that the denominator formula of the GKM Lie superalgebra $\mathcal{G}_1$ is related to the modular form $\Delta_5(\mathbf{Z})$, of $\text{Sp}(2,\mathbb{Z})$, that transforms with character[86]. The modular form, $\Phi_{10}(\mathbf{Z})$, that generates the degeneracy of $\frac{1}{4}$-BPS states is equal to $\Delta_5(\mathbf{Z})^2$. The Weyl-Kac-Borcherds (WKB) denominator formula is a special case of the more general WKB character formula for Lie algebras which gives the characters of integrable highest weight representations of BKM Lie superalgebras[55]. The WKB character formula applied to the trivial representation gives the WKB denominator formula. Let $\mathcal{G}$ be a BKM Lie superalgebra and $\mathcal{W}$ its Weyl group. Let L_+ denote the set of positive roots of the BKM Lie superalgebra and ρ the Weyl vector. Then, the WKB denominator identity for the BKM Lie superalgebra[71] $\mathcal{G}$ is

$$\prod_{\alpha\in L_+} (1-e^{-\alpha})^{\text{mult}(\alpha)} = e^{-\rho} \sum_{w\in\mathcal{W}} (\text{det}w)\ w(e^{\rho} \sum_{\alpha\in L_+} \epsilon(\alpha) e^{\alpha}) \ , \tag{6.17}$$

where $\text{mult}(\alpha)$ is the multiplicity of a root $\alpha \in L_+$[4, 71, 110, 111]. In the above equation, $\det(w)$ is defined to be ± 1 depending on whether w is the product of an even or odd number of reflections and $\epsilon(\alpha)$ is defined to be $(-1)^n$ if α is the sum of n pairwise independent,

[2] The extended S-duality group is defined by including a $\mathbb{Z}_2$ parity operation to the S-duality group $\Gamma_1(N)$. For $N = 1$, this is the group $PGL(2,\mathbb{Z})$[17].

[3] This is equivalent to the Coxeter group generated by the Cartan matrix $A^{(N)}$.

orthogonal imainary simple roots, and 0 otherwise. In the case of BKM Lie superalgebras the roots appear with graded multiplicity – fermionic roots appear with negative multiplicity while bosonic roots appear with positive multiplicity. Following the ideas of Borcherds [4, 71], Gritsenko and Nikulin constructed a superalgebra, $\mathcal{G}_1$ by adding imaginary simple roots – some bosonic and others fermionic. Let us write the Weyl-Kac-Borcherds denominator formula separating it into two parts one of which involves the imaginary simple roots and the other which doesn't, as follows [4]

$$e^{-\pi\imath(\rho,z)} \prod_{\alpha\in L_+} \left(1-e^{-\pi\imath(\alpha,z)}\right)^{\text{mult}(\alpha)} = \left(\sum_{w\in\mathcal{W}} \det(w) \left\{ e^{-\pi\imath(w(\rho),z)} - \sum_{\eta\in M_{II}\cap\mathbb{R}_+\mathcal{M}_{II}} m(\eta)\, e^{-\pi\imath(w(\rho+\eta),z)} \right\}\right) \quad (6.18)$$

where the element $\mathbf{Z} = z_3 f_2 + z_2 f_3 + z_1 f_{-2}$ belongs to the subspace $\mathbb{R}^{2,1} + \imath V^+ \sim \mathbb{H}_2$ obtained upon complexification of the cone V^+. Of the two terms in the sum side, one arises from the real simple roots ($\eta = 0$) and the other arising from the imaginary simple roots ($\eta \neq 0$). The first term thus arises as the sum side of the Lie algebra with no imaginary simple roots. The second term is specific to BKM Lie superalgebras due to the presence of imaginary simple roots with 'multiplicities' $m(\eta) \in \mathbb{Z}$. These multiplicities are determined by the connection with the automorphic form $\Delta_5(\mathbf{Z})$ viz. (6.6). One compares the sum side of the denominator formula to the sum form of the modular form obtained from the additive lift and adds enough imaginary simple roots such that the automorphic properties are attained.

The LHS of (6.17) is identified with the product formula for $\Delta_5(\mathbf{Z})$, and this determines the positive roots L_+ along with their multiplicities– again fermionic roots appear with negative multiplicity in the exponent. However, there is a subtle issue in extracting the multiplicities from the exponent in the product formula – the product formula gives only the difference between the multiplicities of the bosonic and fermionic generators and hence is more like a Witten index. 4Comparing with the denominator identity (6.18), the common factor $q^{1/2}r^{1/2}s^{1/2}$ can be identified with $\exp(-\pi\imath(\rho,z))$ giving us the Weyl vector ρ.

Given the modular form $\Delta_5(\mathbf{Z})$, one can systematically construct the BKM Lie superalgebra $\mathcal{G}_1$ from it. We will illustrate this procedure for the case of the algebra $\mathcal{G}_1$ and the same is used to construct the other algebras that occur in this chapter. Before we summarize

[4]Written here in the notation of Gritsenko and Nikulin, where in particular, one needs to replace ρ by $-\rho$ in Eq. (6.17) (See also section 6.3.1).

the procedure to recognize the algebra, given the denominator identity, it will be useful to list some of the observations that can be made about the expansions of the modular form $\Delta_5(\mathbf{Z})$.

6.3.4 Analyzing the Modular Forms

1. Using the expressions for the real simple roots, $(\delta_1, \delta_2, \delta_3)$ and their inner product with $\mathbf{Z}$, one sees that

$$e^{-\pi i(\delta_1,\mathbf{Z})} = qr \ , \ e^{-\pi i(\delta_2,\mathbf{Z})} = r^{-1} \ \text{ and } e^{-\pi i(\delta_3,\mathbf{Z})} = sr \ .$$

   (Recall that $q = \exp(2\pi i z_1)$, $r = \exp(2\pi i z_2)$ and $s = \exp(2\pi i z_3)$.) Thus, one has $\exp(-\pi i(\rho, \mathbf{Z})) = q^{1/2} r^{1/2} s^{1/2}$. Further, one has the identification relating the root $\alpha[n,\ell,m]$ to $q^n r^\ell s^m$:

$$q^n r^\ell s^m = e^{-\pi i(\alpha[n,\ell,m],\mathbf{Z})} \ ,$$

   where the root $\alpha[n,\ell,m] = n\delta_1 + (-l+m+n)\delta_2 + m\delta_3$ has norm $(2\ell^2 - 8nm)$. The real simple roots are $(\alpha[1,1,0], \alpha[0,-1,0], \alpha[0,1,1])$ and the Weyl vector is $\rho = \alpha[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ in this notation.

2. In the expansion for $\Delta_5(\mathbf{Z})$, all terms (in the expansion given in the Appendix) that arise with coefficient ± 1 arise by the action of all elements of the Weyl group generated by the three real simple roots. They do not involve the imaginary simple roots of the BKM Lie superalgebra. For instance, the terms arising from Weyl reflections associated with the simple real roots of $\mathcal{G}_1$ are

$$(q^{3/2} r^{3/2} s^{1/2}, q^{1/2} r^{-1/2} s^{1/2}, q^{1/2} r^{3/2} s^{3/2}) = q^{1/2} r^{1/2} s^{1/2} (qr, r^{-1}, sr) \ .$$

   Note that we need to pull out an overall factor of $q^{1/2} r^{1/2} s^{1/2}$ in the sum side of the denominator formula to extract the roots.

3. The BKM Lie superalgebra $\mathcal{G}_1$ has an outer S_3 symmetry which permutes the three real simple roots. It is easy to see only the $\delta_1 \leftrightarrow \delta_3$ (or equivalently the $q \leftrightarrow s$) symmetry in the $\Delta_5(\mathbf{Z})$. A formal proof can be given by following Gritsenko and Nikulin's argument for $\mathcal{G}_1$[86, see Prop. 2.1]. Their proof makes use of the non-trivial character v^Γ appearing in the modular transform $\Delta_5(\mathbf{Z})$ (see eq. (5.14).

4. A practical check of the outer S_3 needs us to verify the $\delta_1 \leftrightarrow \delta_2$ invariance of $\Delta_5(\mathbf{Z})$. One can show that under this exchange

$$\alpha[n,\ell,m] \leftrightarrow \alpha[-\ell+m+n,-\ell+2m,m] \ .$$

For instance, the light-like root $\alpha[0,0,1]$ is mapped to another light-like root $\alpha[1,2,1]$. This relates the term $q^{1/2}r^{1/2}s^{3/2}$ to $q^{3/2}r^{5/2}s^{3/2}$ – both have multiplicity -9 in $\Delta_5(\mathbf{Z})$.

Having identified the two sides of the denominator identity with the sum and product representations of the modular form, one can identify the BKM Lie superalgebra that corresponding to the particular modular form as follows. Starting with the product representation of the modular form, and comparing with the above equation, gives us the set of positive roots α of the BKM Lie superalgebra, together with their multiplicities. All multiplicities in the product side are integral as the multiplicities in the product formulae are even integers as discussed earlier. Also, expanding the modular form, we equate the expansion to the sum side (R.H.S) of the denominator formula where each term is thought as coming from the Weyl reflection of a positive root with respect to an element of the Weyl group of the BKM Lie superalgebra. Thus, interpreting the modular form as the denominator formula, we can extract the positive roots and corresponding multiplicities, the set of simple roots, the Weyl group, the Weyl vector and from the above information, the Cartan matrix of the BKM Lie superalgebra.

Before concluding our discussion of $\mathcal{G}_1$, we just emphasize two points: 1)Though it is the modular form $\Phi_{10}(\mathbf{Z})$ that generates the degeneracies of the $\frac{1}{4}$-BPS states, it is the modular form $\Delta_5(\mathbf{Z}) = (\Phi_{10}(\mathbf{Z}))^{1/2}$ that occurs as the denominator of the BKM Lie superalgebra $\mathcal{G}_1$, and 2) One needs both the sum and product representations of the modular forms to compare it with the denominator identity of a BKM Lie superalgebra and reconstruct the algebra from the denominator identity. This concludes our discussion for the BKM Lie superalgebra $\mathcal{G}_1$ coming as the denominator identity of the modular form $\Delta_5(\mathbf{Z})$. Next we look at the case of the families obtained by taking a $\mathbb{Z}_N$-orbifold of the theory giving the various CHL strings.

6.4 The BKM Lie superalgebras $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$

As mentioned in the previous chapter, only for the case of the unorbifolded theory the modular form generating the degeneracy of the $\frac{1}{4}$-BPS states and that generating the R^2 corrections to the effective action are the same. For all the CHL strings generated by taking a $\mathbb{Z}_N$ orbifold, the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$, are related as in eq. (5.4), but different. We will first look at the BKM Lie superalgebras $\widetilde{\mathcal{G}}_N$ corresponding to the modular forms generating the degeneracy of $\frac{1}{4}$-BPS states, i.e. the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$, before going to the BKM Lie superalgebras $\mathcal{G}_N$ related to the modular forms $\Phi_k(\mathbf{Z})$. The method for constructing the BKM Lie superalgebra from the modular forms is along the same lines as discussed for the case of $\Phi_{10}(\mathbf{Z})$. We will now discuss each of the BKM Lie superalgebra $\widetilde{\mathcal{G}}_N$ and $\mathcal{G}_N$ below.

6.4.1 The BKM Lie superalgebra $\widetilde{\mathcal{G}}_2$

The BKM Lie superalgebra $\widetilde{\mathcal{G}}_2$ was constructed by Gritsenko and Nikulin, Cheng and Dabholkar observed that it is the BKM Lie superalgebra corresponding to the modular form $\widetilde{\Delta}_3(\mathbf{Z})$[8] which is the square root of the modular form $\widetilde{\Phi}_6(\mathbf{Z})$ generating the degeneracies of the $\frac{1}{4}$-BPS states in the $\mathbb{Z}_2$ orbifolded theory. $\widetilde{\Delta}_3(\mathbf{Z})$ is also a level 2 modular form with character. The BKM Lie superalgebra $\widetilde{\mathcal{G}}_2$ is given by the Cartan matrix

$$A_{2,II} \equiv \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix} . \tag{6.19}$$

As before, the Cartan matrix is hyperbolic with one negative eigenvalue and rank three. It has four roots which, in the convention introduced above for $\mathcal{G}_1$, are given by the following matrices

$$\delta_1 \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \delta_2 \equiv \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} , \delta_3 \equiv \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} , \delta_4 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix} . \tag{6.20}$$

This can be understood as follows: When one takes the orbifold because of the quantization of the T-duality invariants, due to the presence of the twisted states, not all the splits of the charges in (2.69) are allowed. Instead one has to restrict oneself to the congruence subgroup of $PGL(2,\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = \pm 1, c = 0 \bmod N \right\} / \{\pm 1\} . \tag{6.21}$$

Using the relation between the split of charges, the set of positive real roots relevant for the wall crossing for the $\mathbb{Z}_N$ orbifolded theory are of the form

$$\alpha^{(N)} = \begin{pmatrix} 2n & \ell \\ \ell & 2m \end{pmatrix}, \quad (\alpha, \alpha) = 2, (n, m, \ell) > 0, m \bmod N \ . \tag{6.22}$$

From this, one sees that the two roots $\alpha_1^{(N)} = \delta_1$ and $\alpha_2^{(N)} = \delta_2$ occur for all N. In terms of the variables q, r, s these roots are r^{-1}, qr, qr^3s^2 and s^2r. The extended S-duality group is given by

$$\Gamma_1(2) = \mathcal{W}(A_{2,II}) \rtimes D_2 \ , \tag{6.23}$$

where $\mathcal{W}(A^{(2)})$ is the Weyl group generated by Weyl reflections of all the real simple roots (6.20) and D_2 is the dihedral group that is the symmetry group of the polygon corresponding to the Weyl chamber.

The Weyl vector is given by $\rho = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and is space-like.

6.4.2 The BKM Lie superalgebra $\widetilde{\mathcal{G}}_3$

The BKM Lie superalgebra $\widetilde{\mathcal{G}}_3$ is constructed from the square root of the modular form $\widetilde{\Phi}_4(\mathbf{Z})$, denoted $\widetilde{\Delta}_2(\mathbf{Z})$[8].The BKM Lie superalgebra $\widetilde{\mathcal{G}}_3$ is given by the Cartan matrix

$$A_{3,II} \equiv \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix} . \tag{6.24}$$

In addition to the two real simple roots δ_1 and δ_2, it has 4 other real simple roots which are given by

$$\delta_3 \equiv \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix} , \ \delta_4 \equiv \begin{pmatrix} 4 & 7 \\ 7 & 12 \end{pmatrix} , \ \delta_5 \equiv \begin{pmatrix} 2 & 5 \\ 5 & 12 \end{pmatrix} , \ \delta_6 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} . \tag{6.25}$$

In terms of the variables q, r, s the six roots are $r^{-1}, qr, q^2r^5s^3, q^2r^7s^6, qr^5s^6$ and s^3r.The extended S-duality group in this case is given by

$$\Gamma_1(3) = \mathcal{W}(A_{3,II}) \rtimes D_3 \; . \tag{6.26}$$

The Weyl vector is given by $\rho = \begin{pmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and is space-like.

6.4.3 The BKM Lie superalgebra $\widetilde{\mathcal{G}}_4$

The case of the $\mathbb{Z}_4$ orbifolding is a very interesting one. This is the first example where the N of the orbifolding group is not prime. As we saw in the previous chapter, when we constructed the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ explicitly, the construction of the modular forms for prime N is relatively simpler because the balanced cycle shape conditions give $a_1 = a_N = \frac{24}{N+1}$ and all other $a_r = 0$, leaving no undetermined coefficients. For the case of non-prime N, however, there remain a_r which are not completely fixed by the cycle shape conditions alone and one needs to use other consistency conditions to fix them. The BKM Lie superalgebra $\widetilde{\mathcal{G}}_4$ for $N = 4$ is generated by the modular form $(\widetilde{\Delta}_{3/2}(\mathbf{Z}))^2 = \widetilde{\Phi}_3(\mathbf{Z})$ which was constructed in Chapter 5. Even the BKM Lie superalgebra for the $N = 4$ model is very different in nature to the ones for $N = 1, 2$ and 3. We saw that the BKM Lie superalgebras for $N = 1, 2, 3$ were all of elliptic type with finite volume of the Weyl chambers and had $3, 4$ and 6 real simple roots respectively. The BKM Lie superalgebra $\widetilde{\mathcal{G}}_4$ is of parabolic type with infinite number of real simple roots which is markedly distinct from the $N = 1, 2$ and 3 cases. To write the Cartan matrix of $\widetilde{\mathcal{G}}_4$, let us order the real simple roots into an infinite-dimensional vector as

$$\mathbf{X} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots) = (\ldots, \alpha_1, \beta_{-1}, \alpha_0, \beta_0, \alpha_{-1}, \beta_1, \ldots) \; .$$

Equivalently, let

$$x_m = \begin{cases} \alpha_{-m/2} \; , & m \in 2\mathbb{Z} \\ \beta_{(m-1)/2} \; , & m \in 2\mathbb{Z} + 1 \; . \end{cases} \tag{6.27}$$

The Cartan matrix is given by the matrix of inner products $a_{mn} \equiv \langle x_n, x_m \rangle$ and is given by the infinite-dimensional matrix:

$$A^{(4)} = (a_{nm}) \quad \text{where} \quad a_{nm} = 2 - 4(n-m)^2 \; , \tag{6.28}$$

with $m, n \in \mathbb{Z}$. It is easy to show that the following family of vectors are eigenvectors of the Cartan matrix with zero eigenvalue.

$$\begin{pmatrix} \vdots \\ 1 \\ -3 \\ 3 \\ -1 \\ \vdots \end{pmatrix} \tag{6.29}$$

with $\vdots$ indicating a semi-infinite sequence of zeros. One can show that A has rank three. As usual, the Weyl vector ρ satisfies

$$\langle \rho, x_m \rangle = -1 \ , \ \forall m \ . \tag{6.30}$$

The Weyl vector is given by $\rho = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and is light-like. Recall that the Weyl vectors for the $N = 1, 2$ and 3 theories were space-like.

Let us explicitly write the first eight roots of the infinite number of real simple roots of $\widetilde{\mathcal{G}}_4$ in terms of $PGL(2, \mathbb{Z})$ matrices

$$\begin{gathered} \alpha_0 \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \beta_0 \equiv \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \beta_{-1} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 8 \end{pmatrix} \ , \\ \alpha_1 \equiv \begin{pmatrix} 2 & 7 \\ 7 & 24 \end{pmatrix}, \quad \beta_{-2} \equiv \begin{pmatrix} 6 & 17 \\ 17 & 48 \end{pmatrix} \ , \quad \alpha_{-1} \equiv \begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix} \ , \\ \beta_1 \equiv \begin{pmatrix} 12 & 17 \\ 17 & 24 \end{pmatrix} \ , \quad \alpha_{-2} \equiv \begin{pmatrix} 20 & 31 \\ 31 & 48 \end{pmatrix} \ . \end{gathered} \tag{6.31}$$

In terms of the variables q, r, s these roots are given by

$$r^{-1} \ , \ qr \ , \ rs^4 \ , \ qr^7s^{12} \ , \ q^3r^{17}s^{24} \ , \ q^3r^7s^4 \ , \ q^6r^{17}s^{12} \ , \ q^{10}r^{31}s^{24} \ . \tag{6.32}$$

These results are compatible with expectations based on the walls of marginal stability for the $\mathbb{Z}_4$-orbifold based on Sen's arguments, as we will see below. Before that, however, let us verify that the BKM Lie superalgebra has $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ as its denominator formula.

$D_\infty^{(2)}$-Invariance of $\widetilde{\Delta}_{3/2}(\mathbf{Z})$

Let us see if $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ gives rise to the denominator identity for this BKM Lie superalgebra. We will first show that it contains all the real simple roots that one expects from the study of the walls of marginal stability. Using the definition of the even genus-two theta constants, one can easily prove the following two identities about $\widetilde{\Delta}_{3/2}(\mathbf{Z})$.

1. Let $\mathbf{Z}' = \begin{pmatrix} z_1 & -z_2 \\ -z_2 & z_3 \end{pmatrix}$. Then,

$$\widetilde{\Delta}_{3/2}(\mathbf{Z}') = -\widetilde{\Delta}_{3/2}(\mathbf{Z}) \ . \tag{6.33}$$

   This implies that the modular form is an odd function under $r \to r^{-1}$.

2. $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ is invariant under the exchange $z_1 \leftrightarrow 4z_3$. This implies that the modular form is an odd function under the exchange $q \leftrightarrow s^4$.

Next, the $D_\infty^{(2)}$-generators γ and δ act on the roots x_m written as a 2×2 matrix as follows:

$$\gamma : x_m \longrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \cdot x_m \cdot \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}^{\mathrm{T}} \ , \tag{6.34}$$

$$\delta : x_m \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x_m \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathrm{T}} \ . \tag{6.35}$$

The matrix γ is denoted by $\gamma^{(4)}$ in [8]. $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ is invariant under the symmetry generated by the embedding of γ and δ into $G_0(4) \in Sp(2,\mathbb{Z})$. This implies that under the action of γ and δ,

$$\widetilde{\Delta}_{3/2}(\mathbf{Z}) \to \pm\, \widetilde{\Delta}_{3/2}(\mathbf{Z}) \ .$$

One can show that the sign must be $+1$ by observing that any pair of terms in the Fourier expansion of $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ related by the action of γ (δ resp.) appear with the same Fourier coefficient. For instance, the terms associated with the two simple roots α_0 and β_0 related by the action of δ appear with coefficient $+1$. Similarly, the terms associated with the real simple roots β_0 and β_{-1} related by a γ-translation also appear with coefficient $+1$. Thus, we see that $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ is invariant under the full dihedral group $D_\infty^{(2)}$. This provides an *all-orders* proof that the infinite real simple roots given by the vector $\mathbf{X}$ all appear in the Fourier expansion of $\widetilde{\Delta}_{3/2}(\mathbf{Z})$.

The $q \to s^4$ symmetry of the modular form is equivalent to the symmetry generated by the dihedral generator, y, as defined in Eq. (2.73).

Weyl Transformation of $\widetilde{\Delta}_{3/2}(\mathbf{Z})$

The transformation $r \to r^{-1}$ is the Weyl reflection about the root α_0 and as discussed earlier (see Eq. (6.33)), the modular form is odd under the Weyl reflection. One has

$$w_{\alpha_0} \cdot \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\mathrm{T}} \cdot \mathbf{Z} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{6.36}$$

The reflection due to any other elementary Weyl reflection will also have the same sign. We repeat an argument from the appendix A of [8] to show this. First, the reflection due to α_0 is represented by the matrix $w_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The action on $\mathbf{Z}$ is equivalent to $Sp(2,\mathbb{Z})$ action by the matrix[86]

$$M = \begin{pmatrix} (w_0^{-1})^{\mathrm{T}} & 0 \\ 0 & w_0 \end{pmatrix} ,$$

The minus sign due to the Weyl reflection implies that the character, $v(M)$, associated with the modular form $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ is such that $v(M) = -1$. Next, any other elementary Weyl reflection, w, must be conjugate to w_0 – this is a consequence of dihedral symmetry, $D_\infty^{(2)}$. Hence, one has $w = s \cdot w_0 \cdot s^{-1}$ for some invertible matrix s. It follows that the character associated with the Weyl reflection w is the same as that for w_0. In others, $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ is odd under all elementary reflections. Hence one has

$$\widetilde{\Delta}_{3/2}(w \cdot \mathbf{Z}) = \det(w)\ \widetilde{\Delta}_{3/2}(\mathbf{Z}) . \tag{6.37}$$

We thus see that the extended S-duality group for $N = 4$ is given by[5]

$$\mathcal{W}(A^{(4)}) \rtimes D_\infty^{(2)} , \tag{6.38}$$

where $\mathcal{W}(A^{(4)})$ is the Coxeter group generated by the reflections by all real simple roots x_m and $D_\infty^{(2)}$ is the infinite-dimensional dihedral group generated by γ and δ.

Although the structure of the BKM Lie superalgebra $\widetilde{\mathcal{G}}_4$ is more complicated as compared

[5]The generator y is not realized as an element of a level 4 subgroup of $PGL(2,\mathbb{Z})$ and thus is not an element of the extended S-duality group. This is similar to what happens for $N = 2, 3$[8].

to $\widetilde{\mathcal{G}}_1, \widetilde{\mathcal{G}}_2, \widetilde{\mathcal{G}}_3$, the correspondence between the walls of the Weyl chambers and the walls of marginal stability which was present in the $N = 1, 2, 3$ theories continues to hold even for $N = 4$ and is in accordance with Sen's expectations.

This concludes our study of the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and the corresponding BKM Lie superalgebras $\widetilde{\mathcal{G}}_N$. We now study the modular forms $\Phi_k(\mathbf{Z})$ and the BKM Lie superalgebras $\mathcal{G}_N$ corresponding to them.

6.4.4 The Family of BKM Lie superalgebras $\mathcal{G}_N$

The class of BKM Lie superalgebras $\mathcal{G}_N$ arise from the modular forms $\Phi_k(\mathbf{Z})$. It was first shown in [7] that the modular forms $\Delta_{k/2}(\mathbf{Z})$ are indeed given by the denominator formula for BKM Lie superalgebra $\mathcal{G}_N$, that are closely related to the BKM Lie superalgebra $\mathcal{G}_1$ constructed by Gritsenko and Nikulin from the modular form $\Delta_5(\mathbf{Z})$. In particular, it was shown that

1. All the algebras arise as (different) automorphic corrections to the Lie algebra associated with the rank three Cartan matrix $A_{1,II}$, from which $\mathcal{G}_1$ is also constructed.

2. The real simple roots (and hence the Cartan matrix $A_{1,II}$) for the $\mathcal{G}_N$ are identical to the real roots of $\mathfrak{g}(A_{1,II})$. This implies that the Weyl group is identical as well. This is in contrast to the case for the BKM Lie superalgebras $\widetilde{\mathcal{G}}_N$ where the root system, Cartan matrix, and hence also the Weyl group was different for different N. However, this is consistent because for $N > 1$, this Weyl group is no longer the symmetry group of the lattice of dyonic charges as it was for $N = 1$. The reason is that the lattice of dyonic charges is *not* generated by $1/\Phi_k(\mathbf{Z})$, but instead by $1/\widetilde{\Phi}_k(\mathbf{Z})$.

3. The multiplicities of the imaginary simple roots are, however, different. For instance, imaginary roots of the form $t\eta_0$, where η_0 is a primitive light-like simple root, have a multiplicity $m(t\eta_0)$ given by the formula:

$$1 - \sum_{t\in\mathbb{N}} m(t\eta_0)\; q^n = \prod_{n\in\mathbb{N}} (1-q^n)^{\frac{k-4}{2}} (1-q^{Nn})^{\frac{k+2}{2}}$$

   Note that this formula correctly reproduces the multiplicities of the imaginary roots for $\mathcal{G}_1$ as found by Gritsenko and Nikulin[86].

4. There are also other imaginary simple roots which are not light-like whose multiplicities are determined implicitly by the modular form $\Delta_{k/2}(\mathbf{Z})$.

This completes our discussion of the BKM Lie superalgebras associated to the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$. We conclude with a few comments.

6.5 Conclusion

We have seen that the square root of the modular forms that generate the dyonic degeneracies and the R^2 corrections to the string effective action are related to BKM Lie superalgebras. This is a very interesting result, for the origin of the underlying BKM Lie superalgebra structure to the theory is not immediately apparent. That the degeneracy of BPS states should be given by modular forms, is itself a very remarkable result, for there is no obvious reason that it should have turned out to be so. In particular, the degeneracy of the $\frac{1}{4}$-BPS states are given by Siegel modular forms and it is very remarkable that the degeneracies should be such that they add up exactly to be given by a Siegel modular form.

Another important aspect to note is the dependence of the modular properties on the supersymmetry. The degeneracy of $\frac{1}{2}$-BPS states are given by products of η-functions, while the degeneracies of the $\frac{1}{4}$-BPS states are given by more non-trivial modular forms whose transformation properties are more involved thatn the η functions. Increased amount of supersymmetry seems to play a crucial role in the kind of modular forms that generate the degeneracy of states preserving the supersymmetry.

Also, that the modular forms should be related to BKM Lie superalgebras is an equally non-trivial and remarkable. Again, supersymmetry seems to play an important role in the kind of algebras that are related to the structure. For example the infinite-dimensional Lie algebras related to the genus-one modular forms are the affine Kac-Moody Lie algebras. Requiring $\mathcal{N} = 4$ supersymmetry graduates this to BKM Lie superalgebras which have a far more involved structure than the affine Kac-Moody Lie algebras. The appearance of the BKM Lie superalgebras appears not merely to be incidental, as can be seen from the correspondence between the walls of the Weyl chambers of the BKM Lie superalgebras and the walls of marginal stability of the $\frac{1}{4}$-BPS states, and seem to contain information about the CHL theory they come from. It will be interesting to explore this direction further to unearth more connections between the family of BKM Lie superalgebras and the CHL strings. Also, one can ask if such structures exist for other models. These are all new and interesting directions in which one can look at. Harvey and Moore have considered the algebra of BPS states [112, 113]. It is of interest to ask whether the BKM Lie superalgebrathat we have found have any relation to the algebra of BPS states.

7
Results of the Thesis

In this chapter we list the set of results in this thesis that are due to the author of the thesis, obtained as part of work done with collaborators. These results have been presented in [7, 9, 10]. We list the results along with the context in which they were worked.

- **In [7] the existence of a family of BKM Lie superalgebras, $\mathcal{G}_N$, were shown whose Weyl-Kac-Borcherds denominator formula gives rise to a genus-two modular form at level N, $\Delta_{k/2}(\mathbf{Z})$, for $(N,k) = (1,10)$, $(2,6)$, $(3,4)$, $(4,\frac{3}{2})$ and $(5,2)$.**

  Let us briefly recall, from the previous chapters, the context of the above result. Starting with the work of Dijkgraaf, Verlinde, and Verlinde[1] it was found by Jatkar and Sen that the generating function of the degeneracies of $\frac{1}{4}$-BPS states in a class of $\mathcal{N}=4$ supersymmetric string theories in four space-time dimensions[6], was given by genus-two Siegel modular forms, denoted $\widetilde{\Phi}_k(\mathbf{Z})$[2]. It was also observed by DVV that the 'square root' of the modular form in question appears as the denominator identity of a BKM Lie superalgebra. In [7] this idea was extended to the family of modular forms constructed by Jatkar and Sen. A family of BKM Lie superalgebras were constructed, along the lines of the work by Gritsenko and Nikulin[86], whose denominator identities were given by square roots, $\Delta_{k/2}(\mathbf{Z})$, of the genus-two modular forms $\Phi_k(\mathbf{Z})$ generating the R^2-corrections to the string effective action. All the BKM Lie superalgebras are given by the Cartan matrix

$$A_{1,II} \equiv \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} . \tag{7.1}$$

The algebras, denoted $\mathcal{G}_N$, have three real simple roots given by the following $PGL(2,\mathbb{Z})$ matrices

$$\delta_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \delta_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ , \ \delta_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \ . \tag{7.2}$$

The Weyl group is generated by the three elementary reflections, (w_1, w_2, w_3), with respect to the three real simple roots. It is given by[86] (see also [106, 17])

$$PGL(2,\mathbb{Z}) = \mathcal{W}(A_{1,II}) \rtimes S_3 \ , \tag{7.3}$$

where S_3 is the group of permutations of the three real simple roots. The $\mathcal{G}_N$ also have a lattice Weyl vector that satisfies

$$(\rho, \delta_i) = -\frac{(\delta_i, \delta_i)}{2} = -1 \tag{7.4}$$

with all the real simple roots.

The Cartan matrix $A_{1,II}$, the set of real simple roots, $\delta_1, \delta_2, \delta_3$, the Weyl group $\mathcal{W}(A_{1,II})$, the fundamental Weyl chambers and the lattice Weyl vector ρ of the BKM Lie superalgebras $\mathcal{G}_N$ do not change with the orbifolding group $\mathbb{Z}_N$. All the algebras arise as (different) automorphic corrections to the Lie algebra associated with the rank three Cartan matrix $A_{1,II}$ with real simple roots given in (7.2). The BKM Lie superalgebras $\mathcal{G}_N$ also have imaginary roots whose norm, with respect to a given inner product in the root space, is not positive definite, i.e. the norm $(\eta, \eta) \leq 0$. The set of imaginary roots of the BKM Lie superalgebras $\mathcal{G}_N$ also do not change with N. Their multiplicities, however, change with N. We discuss this point next.

- **It was shown in [7] that the multiplicities of the imaginary simple roots for the BKM Lie superalgebras $\mathcal{G}_N$ are different for different N. The primitive light-like simple roots $t\eta_0$ have a multiplicity $m(t\eta_0)$ given by the formula:**

$$1 - \sum_{t\in\mathbb{N}} m(t\eta_0)\ q^n = \frac{\sqrt{g_\rho(\tau)}}{\eta(\tau)^3}$$

As mentioned above, the Cartan matrix, Weyl group, and the set of real and imaginary simple roots for the $\mathcal{G}_N$ remain the same for all values of N. The modular forms leading to these algebras, and hence the denominator identities of the algebras, however, are different from each other. The difference in the denominator identities is in the coefficients of the terms occurring in the expansion, whereas the terms themselves undergo no change. The generating functions of the multiplicity factors of the various multiples of the form $t\eta$, of the light-like simple roots η, for different values of N are given in terms of a single formula

$$\theta_1(\tau, z) \left(1 - \sum_{t\in\mathbb{N}} m(t\eta_0)\ q^t\right) = \psi_{k/2,1/2}(\tau, z)\ . \tag{7.5}$$

From the above we see the pattern in the progression of the $m(\eta_0)$ as the orbifolding group $\mathbb{Z}_N$ varies. For example, for $N = 1$ the formula reproduces the result obtained by Gritsenko and Nikulin[86]

$$1 - \sum_{t\in\mathbb{N}} m(t\eta_0)q^t = \prod_{k\in\mathbb{N}}(1 - q^k)^9 = \frac{\sqrt{f^{(10)}(\tau)}}{\eta(\tau)^3} \tag{7.6}$$

where the multiplicity of the light-like roots is 9, while for $N = 2$, it gives

$$1 - \sum_{t\in\mathbb{N}} m(t\eta_0)\ q^t = \prod_{k\in\mathbb{N}}(1 - q^k)(1 - q^{2k})^4 = \frac{\sqrt{f^{(6)}(\tau)}}{\eta(\tau)^3} \tag{7.7}$$

with the multiplicity of the light-like roots being 4.

This is similar to with the twisted denominator formula of Niemann[80] where the sub-algebras are obtained by the orbifolding action on the fake Monster Lie algebra.

- **In [7] the modular properties of the modular forms $\Delta_k(\mathbf{Z})$ generating the BKM Lie superalgebras $\mathcal{G}_N$ of the $\mathbb{Z}_N$ orbifolded CHL strings.**

  The modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ generate, respectively, the degeneracy of the $\frac{1}{4}$-BPS states and the R^2 corrections to the string effective action in the CHL orbifolds. As explained in the pervious chapter, its the modular forms $\Delta_k(\mathbf{Z})$ and $\widetilde{\Delta}_k(\mathbf{Z})$, that are the 'square roots' of the modular forms $\Phi_k(\mathbf{Z})$ and $\widetilde{\Phi}_k(\mathbf{Z})$ respectively, that occur as the denominator identities of BKM Lie superalgebras. The modular form $\Delta_5(\mathbf{Z})$ was found to be the denominator identity of the BKM Lie superalgebra $\mathcal{G}_1$ by Gritsenko and Nikulin. In [7] the modular forms that occur as the denominator identities of the family of BKM Lie superalgebras $\mathcal{G}_N$ were constructed. However, one needs to check the modular properties of the $\Delta_k(\mathbf{Z})$ before interpreting them as the denominator of a BKM Lie superalgebra. In [7] the modular properties of the modular forms $\Delta_k(\mathbf{Z})$ were shown, and the modular forms constructed from additive lifts of Jacobi forms with half-integer index. Some of the modular forms $\Delta_k(\mathbf{Z})$ were also given as products of even genus-two theta constants.

- **In [9] the procedure to construct the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ for the case of general non-prime N, of the orbifolding group $\mathbb{Z}_N$, of the CHL string was given. In particular, the modular forms $\widetilde{\Phi}_3(\mathbf{Z})$ generating the degeneracy of $\frac{1}{4}$-BPS states, and $\Phi_3(\mathbf{Z})$ generating the string R^2 corrections in the $\mathbb{Z}_4$ orbifolded CHL theory were explicitly constructed, in the sum and product forms, and studied.**

  Jatkar and Sen had constructed the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ for the case of prime N for the orbifolding group $\mathbb{Z}_N$[2]. For the case of composite N, however, only the general behavior was subsequently studied[30]. In [9] the modular forms $\widetilde{\Phi}_3(\mathbf{Z})$ and $\Phi_3(\mathbf{Z})$ were explicitly constructed in the sum form via the additive lift. Further, the systematics of the product formulae were worked out and explicitly computed for the case of $N = 4$. Also, the general procedure to construct the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ and $\Phi_k(\mathbf{Z})$ for the case of general non-prime N was given, thereby completing the construction of the genus-two Siegel modular forms for all $\mathbb{Z}_N$ orbifolds of the CHL strings.

  The product form of $\Phi_3(\mathbf{Z})$ is given by[32]

$$\Phi_3(\mathbf{Z}) = qrs \prod_{(n,\ell,m)} \Big(1 - q^n r^\ell s^m\Big)^{\hat{c}^0 - \hat{c}^2} \times \Big(1 - \big(q^n r^\ell s^m\big)^2\Big)^{\hat{c}^2 - \hat{c}^1} \times \Big(1 - \big(q^n r^\ell s^m\big)^4\Big)^{\hat{c}^1} \ . \quad (7.8)$$

  The $\hat{c}^a(n,\ell)$ are given as the Fourier coefficients of

$$\widehat{F}^a(z_1, z_2) = \sum_{b=0}^{3} F^{a,b}(z_1, z_2) \ , \quad (7.9)$$

  where $F^{(a,b)}(z_1, z_2)$ are the twisted elliptic genera for a $\mathbb{Z}_N$-orbifold of $K3$ given as:

$$F^{r,s}(z_1, z_2) = \tfrac{1}{N} \mathrm{Tr}_{RR,g^r}\Big((-)^{F_L + F_R} g^s q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi\imath z F_L}\Big) \ , \quad 0 \le r, s \le (N-1) \quad (7.10)$$

  and g generates $\mathbb{Z}_N$ and $q = \exp(2\pi\imath z_1)$. For $N = 4$, the various twisted elliptic genera are given [9]by

$$\begin{aligned}
F^{0,0}(z_1,z_2) &= 2A(z_1,z_2) \\
F^{0,1}(z_1,z_2) &= F^{0,3}(z_1,z_2) = \tfrac{1}{3}A(z_1,z_2) + \Big[-\tfrac{1}{12}E_2(z_1) + \tfrac{1}{2}E_4(z_1)\Big]B(z_1,z_2) \\
F^{0,2}(z_1,z_2) &= \tfrac{2}{3}A(z_1,z_2) + \tfrac{1}{3}E_2(z_1)B(z_1,z_2) \\
F^{1,k}(z_1,z_2) &= F^{3,3k}(z_1,z_2) = \tfrac{1}{3}A(z_1,z_2) + \Big[-\tfrac{1}{24}E_2\big(\tfrac{z_1+k}{2}\big) + \tfrac{1}{8}E_4\big(\tfrac{z_1+k}{4}\big)\Big]B(z_1,z_2) \\
F^{2,2k}(z_1,z_2) &= \tfrac{2}{3}A(z_1,z_2) - \tfrac{1}{6}E_2\big(\tfrac{z_1+k}{2}\big)B(z_1,z_2) \\
F^{2,2k+1}(z_1,z_2) &= \tfrac{1}{3}A(z_1,z_2) + \Big[\tfrac{5}{12}E_2(z_1) - \tfrac{1}{2}E_4(z_1)\Big]B(z_1,z_2)
\end{aligned} \tag{7.11}$$

and

$$\begin{aligned}
\widehat{F}^0(z_1,z_2) &= \tfrac{10}{3}A(z_1,z_2) + \tfrac{1}{6}E_2(z_1)B(z_1,z_2) + E_4(z_1)B(z_1,z_2) \\
\widehat{F}^1(z_1,z_2) &= \tfrac{4}{3}A(z_1,z_2) + \tfrac{1}{6}E_2(z_1)B(z_1,z_2) - \tfrac{1}{2}E_4(z_1)B(z_1,z_2) \\
\widehat{F}^2(z_1,z_2) &= 2A(z_1,z_2) + \tfrac{1}{2}E_2(z_1)B(z_1,z_2) - E_4(z_1)B(z_1,z_2)\ .
\end{aligned} \tag{7.12}$$

The product formula for $\widetilde{\Phi}_3(\mathbf{Z})$ is

$$\widetilde{\Phi}_3(\mathbf{Z}) = q^{1/4} r s \prod_{a}^{3} \prod_{\substack{\ell,m\in\mathbb{Z},\\ n\in\mathbb{Z}+\frac{a}{4}}} \Big(1 - q^n r^\ell s^m\Big)^{\sum_{b=0}^{3} \omega^{-bm} c^{(a,b)}(4nm-\ell^2)} \tag{7.13}$$

where $\omega = \exp(\frac{2\pi\imath}{3})$ is a cube root of unity, and $c^{(a,b)}(4nm-\ell^2)$ are the Fourier coefficients of the twisted elliptic genera, $F^{(a,b)}(z_1,z_2)$.

It has also been shown in [9] that the $N=4$ modular forms can be written as the square of the product of three even genus-two theta constants. One has for $\Phi_3(\mathbf{Z})$:

$$\Phi_3(\mathbf{Z}) = \left(\frac{1}{8}\,\theta\begin{bmatrix}1\\0\\0\\1\end{bmatrix}(2\mathbf{Z})\ \theta\begin{bmatrix}0\\1\\1\\0\end{bmatrix}(2\mathbf{Z})\ \theta\begin{bmatrix}1\\1\\1\\1\end{bmatrix}(2\mathbf{Z})\right)^2 \equiv \big[\Delta_{3/2}(\mathbf{Z})\big]^2\ . \tag{7.14}$$

This is a known modular form with character of weight three at level four. For instance, see Aoki-Ibukiyama[97], where this is called f_3. For the case of $\widetilde{\Phi}_3(\mathbf{Z})$ one has:

$$\widetilde{\Phi}_3(\mathbf{Z}) = \left(\frac{1}{4}\,\theta\begin{bmatrix}0\\0\\1\\1\end{bmatrix}(\mathbf{Z}')\ \theta\begin{bmatrix}1\\1\\0\\0\end{bmatrix}(\mathbf{Z}')\ \theta\begin{bmatrix}1\\1\\1\\1\end{bmatrix}(\mathbf{Z}')\right)^2 \equiv \Big[\widetilde{\Delta}_{3/2}(\mathbf{Z})\Big]^2\ . \tag{7.15}$$

where $\mathbf{Z}' = \begin{pmatrix} \frac{1}{2}z_1 & z_2 \\ z_2 & 2z_3 \end{pmatrix}$.

We discuss the additive lift for the modular forms $\widetilde{\Phi}_3(\mathbf{Z})$ and $\Phi_3(\mathbf{Z})$ in the next point where the η-products are discussed.

- **The generating function of $\frac{1}{2}$-BPS states is given by multiplicative η-products. In [9] the η-products for *all* groups that arise as symplectic involutions of $K3$ were given extending results due to Dummit, Kisilevsky and McKay[93] as well as Mason[94]. In addition, the conjecture for constructing the additive lift leading to the Siegel modular forms $\Phi_k(\mathbf{Z})$ and $\widetilde{\Phi}_k(\mathbf{Z})$ when $G = \mathbb{Z}_N$ for all N, in terms of the multiplicative η-products, was proposed and the modular forms $\Phi_3(\mathbf{Z})$ and $\widetilde{\Phi}_3(\mathbf{Z})$ constructed from it.**

  The generating function for the degeneracies $d(n)$ of $\frac{1}{2}$-BPS states for the $\mathbb{Z}_N$ CHL orbifold, taking into account that the electric charge is quantized such that $N\mathbf{q}_e^2 \in 2\mathbb{Z}$, is given as:

$$\frac{16}{g_\rho(\tau/N)} \equiv \sum_{n=-1}^{\infty} d(n)\ q^{n/N}\ . \tag{7.16}$$

  It was shown that an ansatz for $g_\rho(\tau)$ in the form of an η-product

$$g_\rho(\tau) = \prod_{r=1}^{N} \eta(r\tau)^{a_r} = \eta(\tau)^{a_1}\eta(2\tau)^{a_2}\cdots\eta(N\tau)^{a_N}\ . \tag{7.17}$$

  with balanced cycle shapes satisfying certain additional conditions gives the correct degeneracy for the $\frac{1}{2}$-BPS states in all CHL models where the cycle shapes arise from the action of Nikulin involutions on $H^*(K3)$ (in the dual description of the CHL orbifold as asupersymmetric orbifold of type II string theory on $K3 \times T^2$), including product groups such as $\mathbb{Z}_M \times \mathbb{Z}_N$.

  The modular form $g_\rho(\tau)$, of weight $(k+2)$, satisfies the following conditions:

  1. The coefficients a_r satisfy

$$\begin{aligned} \left(Na_1 + N\tfrac{a_2}{2} + \cdots + a_N\right) &= 24\ , \\ a_1 + a_2 + \cdots + a_N &= 2(k+2)\ , \\ \left(1^{a_1}2^{a_2}\cdots N^{a_N}\right)^{-1} &= \mathrm{vol}^{\perp}\ , \end{aligned} \tag{7.18}$$

where $\text{vol}^\perp$ be the volume of the unit cell in $\Gamma^\perp$.

2. The only permitted cycles are of length r such that $r|N$, and hence $a_r = 0$ unless $r|N$. Thus, when N is prime, only a_1 and a_N are non-zero which agrees with known results.

3. The requirement that the cycle be balanced implies that $a_1 = a_N$ among other things. It also implies that the first equation in Eq. (5.22) can be rewritten as

$$a_1 + 2a_2 + \cdots + N a_N = 24 \tag{7.19}$$

This connects to the results of Dummit, Kisilevsky and McKay[93] on the multiplicative balanced cycle shapes of elements of M_{24}[92].

Also, it was conjectured in [9] that the Jacobi form of weight k, index 1 and level N that is the seed for the additive (Maaß) lift leading to the Siegel modular form $\Phi_k(\mathbf{Z})$ when $G = \mathbb{Z}_N$ for all N is given by $\frac{\vartheta_1(z_1,z_2)^2}{\eta(z_1)^6}\, g_\rho(z_1)$. The additive lift giving the modular forms $\Phi_k(\mathbf{Z})$ as an infinite sum is given as

$$\phi_{k,1}(z_1, z_2) = \frac{\vartheta_1(z_1, z_2)^2}{\eta(z_1)^6}\, g_\rho(z_1) = \sum_{n,\ell} a(n,\ell)\, q^n r^\ell \ . \tag{7.20}$$

A similar additive lift for $\widetilde{\Phi}_k(\mathbf{Z})$ is given by the following additive seed:

$$\widetilde{\phi}_{k,1}(z_1, z_2) = \frac{\vartheta_1(z_1, z_2)^2}{\eta(z_1)^6}\, g_\rho(z_1/N) \ . \tag{7.21}$$

- **The BKM Lie superalgebra for $\Delta_{3/2}(\mathbf{Z})$ is shown to be similar to the ones appearing in [7]. The Cartan matrix, Weyl vector and Weyl group remain unchanged by the orbifolding. However, the multiplicities of the imaginary simple root do depend on the orbifolding.The BKM Lie superalgebra for $\widetilde{\Delta}_{3/2}(\mathbf{Z})$ is of parabolic type with infinite real simple roots (labelled by an integer) with Cartan matrix**

$$A^{(4)} = (a_{nm}) \quad \textbf{where} \quad a_{nm} = 2 - 4(n-m)^2 \ , \tag{6.28}$$

and a light-like Weyl vector. The walls of marginal stability for the $N = 4$ model get mapped to the walls of the fundamental Weyl chamber of the

BKM Lie superalgebra.

We looked at the BKM Lie superalgebras underlying the degeneracy of the $\frac{1}{4}$-BPS states in the family of CHL strings in the previous chapters. The BKM Lie superalgebra, $\mathcal{G}_1$, for the $N = 1$ case was constructed by Gritsenko and Nikulin [86] and was extended to other values of N in [7] and [8]. As mentioned above the BKM Lie superalgebras $\mathcal{G}_N$ corresponding to the modular forms $\Phi_k(\mathbf{Z})$ for $N = 2, 3$ and 5 were constructed in [7], while the BKM Lie superalgebras $\widetilde{\mathcal{G}}_N$ corresponding to the modular forms $\widetilde{\Phi}_k(\mathbf{Z})$ for $N = 2, 3$ were constructed by Gritsenko and Nikulin [108] [109] and their relation to the degeneracies of $\frac{1}{4}$-BPS states of the CHL models for $N = 2, 3$ was pointed out by Cheng and Dabholkar [8].

It was predicted in [8] that the BKM Lie superalgebra for $N > 3$ CHL models may not exist but since the modular forms $\Phi_3(\mathbf{Z})$ and $\widetilde{\Phi}_3(\mathbf{Z})$ corresponding to the $N = 4$ model had not been explicitly constructed before [7] it could not be verified. In [7] the BKM Lie superalgebra underlying the degeneracy of the $\frac{1}{4}$-BPS states in the $N = 4$ CHL model was shown to exist, and was constructed from the corresponding modular forms.

The BKM Lie superalgebra $\widetilde{\mathcal{G}}_4$ for $N = 4$ is generated by the modular form $(\widetilde{\Delta}_{3/2}(\mathbf{Z}))^2 = \widetilde{\Phi}_3(\mathbf{Z})$. The BKM Lie superalgebra algebra $\widetilde{\mathcal{G}}_4$ is of parabolic type with infinite number of real simple roots. To write the Cartan matrix of $\widetilde{\mathcal{G}}_4$, let us order the real simple roots into an infinite-dimensional vector as

$$\mathbf{X} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots) = (\ldots, \alpha_1, \beta_{-1}, \alpha_0, \beta_0, \alpha_{-1}, \beta_1, \ldots) \ .$$

Equivalently, let

$$x_m = \begin{cases} \alpha_{-m/2} \ , & m \in 2\mathbb{Z} \\ \beta_{(m-1)/2} \ , & m \in 2\mathbb{Z}+1 \ . \end{cases} \tag{7.22}$$

The Cartan matrix is given by the matrix of inner products $a_{mn} \equiv \langle x_n, x_m \rangle$ and is given by the infinite-dimensional matrix:

$$A^{(4)} = (a_{nm}) \quad \text{where} \quad a_{nm} = 2 - 4(n-m)^2 \ , \tag{7.23}$$

with $m, n \in \mathbb{Z}$. It is easy to show that the following family of vectors are eigenvectors of the Cartan matrix with zero eigenvalue.

$$\begin{pmatrix} \vdots \\ 1 \\ -3 \\ 3 \\ -1 \\ \vdots \end{pmatrix} \tag{7.24}$$

with $\vdots$ indicating a semi-infinite sequence of zeros. One can show that A has rank three. As usual, the Weyl vector ρ satisfies

$$\langle \rho, x_m \rangle = -1 \ , \ \forall m \ . \tag{7.25}$$

The Weyl vector is given by $\rho = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and is light-like.

The extended S-duality group for $N = 4$ is given by

$$\mathcal{W}(A^{(4)}) \rtimes D_\infty^{(2)} \ , \tag{7.26}$$

where $\mathcal{W}(A^{(4)})$ is the Coxeter group generated by the reflections by all real simple roots x_m and $D_\infty^{(2)}$ is the infinite-dimensional dihedral group generated by γ and δ which act on the roots x_m written as a 2×2 matrix as follows:

$$\gamma : x_m \longrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \cdot x_m \cdot \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}^{\mathrm{T}} \ , \tag{7.27}$$

$$\delta : x_m \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x_m \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathrm{T}} \ . \tag{7.28}$$

Also, the walls of the Weyl chamber of the BKM Lie superalgebra $\widetilde{\mathcal{G}}_4$ was studied and found to be compatible with Sen's expectations. The fundamental domain/Weyl chamber for $N = 4$ is bounded by an infinite number of semi-circles as the BKM Lie superalgebra has infinite real simple roots. Each of the semi-circles represents a real simple root. The point $\frac{1}{2}$ is approached as a limit point of the infinite sequence of semi-circles.

Starting from the product expansion for $\Delta_{3/2}(\mathbf{Z})$ the BKM Lie superalgebra, $\mathcal{G}_4$ corresponding to it was also constructed in [9]. The Weyl vector ρ is the same as for the algebras $\mathcal{G}_N$ for $N = 1, 2, 3, 5$ for prime N. Also the three real simple roots remain unchanged as before[7]. The imaginary roots remain unchanged as well, but their multiplicities are changed by the orbifolding. For $\Delta_{k/2}(\mathbf{Z}) = (\Phi_k(\mathbf{Z}))^{1/2}$ for prime N, we recall that the BKM Lie superalgebras $\mathcal{G}_N$ were all given by the same Cartan matrix, and had the same set of real simple roots, Weyl group, Weyl vector, and imaginary roots. The orbifolding only changed the multiplicities of the imaginary roots for different values of N. It was seen that the same pattern continues to hold for the BKM Lie superalgebra even when N is non-prime for $\Delta_{3/2}(\mathbf{Z})$.

- **In [10] it has been shown that the counting of $\frac{1}{2}$-BPS states is given by multiplicative η-quotients that are associated with the frame shapes $\tilde{\rho}$ given in Table 5.4, generalizing the corresponding result for CHL strings where the generating functions for the $\frac{1}{2}$-BPS states were given by multiplicative η-products corresponding to cycle shapes.**

  It was shown in [9] that the degeneracy of the electrically charged $\frac{1}{2}$-BPS states are given by multiplicative η-products. The idea was extended to the type II models where the degeneracy of the electrically charged $\frac{1}{2}$-BPS states were shown to be given by multiplicative η-quotients determined by the frame shapes associated with the conjugacy classes of Co_1. Using the modular forms generating the degeneracy of the $\frac{1}{2}$-BPS states, the additive lift for the modular forms generating the degeneracy of the $\frac{1}{4}$-BPS states are constructed for $N = 2, 3$ and a conjecture is provided for $N = 4$.

- **In [10] the modular forms generating the degeneracy of $\frac{1}{4}$-BPS states in the type II models have been found in terms of the modular forms generating the degeneracy of $\frac{1}{4}$-BPS states in the CHL models. A similar relation has also been found for the modular forms generating the string R^2 corrections**

  David, Jatkar and Sen have provided product formulae for the $N = 2, 3$ type II models[31] in terms of the twisted elliptic genus for T^4. In [10] these modular forms have been expressed in terms of the various Siegel modular forms occuring in the CHL models. The modular forms in the CHL models have been well studied and have been interpreted as the denominator identities of BKM Lie superalgebras. Expressing the modular forms of the type II models in terms of the ones occuring in the CHL models should help in studying the underlying BKM Lie superalgebra structure, if any.

- **In [10] a general discussion on the BKM Lie superalgebras corresponding to the the type II models is presented. Though the BKM Lie superalgebras for these models have not been constructed in [10], based on the properties that are expected of these algebras, general directions for finding these algebras, if they exist, has been discussed.**

  The CHL models have been found to have an underlying BKM Lie superalgebra structure to the degeneracy of the $\frac{1}{4}$-BPS states. A natural question to consider would be if such an algebraic structure exists even for the type II models. The modular forms apearing in the type II models seem to have a complicated structure, which does not immediately have the interpretation of a BKM Lie superalgebra. However, since these modular forms can be expressed in terms of the modular forms of the CHL models, which have a BKM Lie superalgebra interpretation, one can guess the properties that a BKM Lie superalgebra, if it exits, is expected to have. A discussion on the same is provided in [10].

8
Conclusion and Future Directions

In this thesis we have studied the various aspects of the counting of dyonic states in string theory. The problem of counting has been of much interest because of the rich mathematical structure underlying it. The degeneracy of the dyonic states are given by modular forms and this strongly suggests the presence of a deeper mathematical structure to the theory. In the words of Barry Mazur, "Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist". That the degeneracy of dyonic states should be such as to be given by such special functions that have very restricted transformation properties and are sensitive to the smallest of perturbations to their structure, seems to be extraordinary.

Equally extrodinary is the appearance of BKM Lie superalgebras related to the modular forms. BKM Lie superalgebras are infinite-dimensional Lie algebras which are very complicated and rich generalizations of classical semi-simple Lie algebras. That they should appear as an underlying symmetry of the degeneracy of the dyonic states is very remarkable. Also, as was mentioned previously, the relation to supersymmetry is another intriguing aspect. The degeneracy of states preserving higher degree of supersymmetry are given by more complicated modular forms and have more involved infinite-dimensional Lie algebra structure underlying them. This seems to indicate an important role for supersymmetry in leading to the modular structure of the generating functions of dyonic degeneracies. A complete understanding of the whole structure, however, is far from apparent at this point. The origin of the various mathematical structures, their significance and implication to the theory are areas that will be very interesting to understand and unearth.

As we have seen, the BKM Lie superalgebra structure undergoes a distinct change in going from $N = 3$ to $N = 4$ in the CHL models. The BKM Lie superalgebra structure for the $N = 4$ case was a parabolic algebra and previously not constructed. It has an infinite number of real simple roots. The BKM Lie superalgebra structure for CHL models with $N > 4$, if they exist, could be much more complicated.

Other areas of future interest are obtaining a better understanding of the BKM Lie superalgebras related to the type II models, and more generally models which come from frame shapes rather than cycle shapes. Also understanding the models with product groups of the form $\mathbb{Z}_N \times \mathbb{Z}_M$, starting from the cycle shapes and generating the modular forms generating the dyonic degeneracy will go towards completing the construction of degeneracy formulas for all orbifoldings of the CHL strings (see [114]). Sen et.al. have constructed the partition functions for torsion > 1 dyons in heterotic string theory on T^6. Seeing if a BKM Lie superalgebra structure exists for these models will extend the construction of [7, 9, 8].

Also, the idea of understanding the degeneracy of $\frac{1}{2}$-BPS states from the symplectic automorphisms of the $K3$ surface (for the case of CHL strings) or from the conjugacy classes of Co_1 (for the case of type II models) is an interesting result which gives a geometric understanding to the origin of these degeneracies. Garbagnati and Sarti have studied symplectic (Nikulin) involutions of $K3$ manifolds[115, 96]. In particular, they have explicitly constructed elliptic $K3$s whose automorphism groups are the Nikulin involutions. Further, they have provided an explicit description of the invariant lattice and its complementary lattice. We anticipate that these results might be relevant in improving our physical understanding the role of the roots of the BKM Lie superalgebras. The Jatkar-Sen construction holds for $N = 11$ as well and it leads to a modular function (i.e., one of weight $k = 0$) $\Phi_0(\mathbf{Z})$ and it is believed that a CHL string may exist. In the type IIA picture, the $\mathbb{Z}_{11}$ is no longer a symplectic Nikulin involution, it acts non-trivially on $H^*(K3)$ and not on $H^{1,1}(K3)$ alone. It is of interest to study aspects of the $\mathbb{Z}_N$ orbifold both from the physical and mathematical point of view.

As we have seen, for affine Kac-Moody algebras, the presence of light-like imaginary roots in L_+ leads to powers of the Dedekind eta function appearing in the product form of the Weyl-Kac denominator formula. As is well known, $q^{1/24}/\eta(\tau)$ is the generating function of partitions of n (equivalently, Young diagrams with n boxes). An interesting generalisation is the generating function of plane partitions (or 3D Young diagrams) has a nice product representation $\eta_{3D} \sim \prod_n (1-q^n)^n$ (due to MacMahon). This function appears in the counting of D0-branes in the work of Gopakumar-Vafa[116, 117]. Is there an algebraic interpretation for this? The addition of D2-branes to this enriches this story and leads to interesting formulae[118].

One can also carry out a similar programme for models with $\mathcal{N} = 2$ supersymmetry[119, 120, 121]. As mentioned before, the high degree of supersymmetry makes all the beautiful mathematical structure highly symmetry specific. Our ultimate aim is to understand the microscopic description of general black holes. For this it is necessary to understand the above ideas when the degree of symmetry of the system is reduced. Starting with $\mathcal{N} = 2$ models is a good way to finally graduating to the general case.

A

Theta functions

A.1 Genus-one theta functions

The genus-one theta functions are defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z_1, z_2) = \sum_{l \in \mathbb{Z}} q^{\frac{1}{2}(l+\frac{a}{2})^2} \; r^{(l+\frac{a}{2})} \; e^{i\pi l b} \; , \tag{A.1}$$

where $a.b \in (0,1) \bmod 2$. One has $\vartheta_1(z_1, z_2) \equiv \theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z_1, z_2)$, $\vartheta_2(z_1, z_2) \equiv \theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](z_1, z_2)$, $\vartheta_3(z_1, z_2) \equiv \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z_1, z_2)$ and $\vartheta_4(z_1, z_2) \equiv \theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](z_1, z_2)$.

The transformations of $\vartheta_1(\tau, z)$ under modular transformations is given by

$$\begin{aligned} T: \quad \vartheta_1(\tau+1, z) &= e^{i\pi/4} \; \vartheta_1(\tau, z) \; , \\ S: \quad \vartheta_1(-1/\tau, -z/\tau) &= -\frac{1}{q^{1/2} r} \; e^{\pi i z^2/\tau} \; \vartheta_1(\tau, z) \; , \end{aligned} \tag{A.2}$$

with $q = \exp(2\pi i \tau)$ and $r = \exp(2\pi i z)$.
The Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = e^{2\pi i \tau/24} \prod_{n=1}^{\infty} (1 - q^n) \; . \tag{A.3}$$

The transformation of the Dedekind eta function under the modular group is given by

$$\begin{aligned} T: \quad \eta(\tau+1) &= e^{\pi i/12} \; \eta(\tau) \; , \\ S: \quad \eta(-1/\tau) &= e^{-\pi i/4} \; (\tau)^{1/2} \; \eta(\tau) \; . \end{aligned} \tag{A.4}$$

The transformation of $\eta(N\tau)$ is given by

$$\begin{aligned} T: \quad \eta(N\tau+N) &= e^{N\pi i/12}\,\eta(\tau)\ , \\ S: \quad \eta(-1/\tau) &= \frac{e^{-\pi i/4}}{\sqrt{N}}\,(\tau)^{1/2}\,\eta(\tau/N)\ . \end{aligned} \tag{A.5}$$

One can see that $\eta(N\tau)$ transforms into $\eta(\tau/N)$ under the S transformation. $\eta(N\tau)$ gets mapped to itself only under the subgroup, $\Gamma_0(N)$ of $SL(2,\mathbb{Z})$. Following Niemann[80], let

$$\psi_j(\tau) \equiv \eta\left(\tfrac{\tau+j}{N}+j\right)\ ,\ j=0,1,\ldots,N-1 \bmod N\ . \tag{A.6}$$

Both S and T no longer have a diagonal action on the $\psi_j(\tau)$. One has

$$T: \quad \psi_j(\tau+1) = e^{\pi i/12}\,\psi_{j+1}(\tau) \tag{A.7}$$

$$S: \quad \psi_j(-1/\tau) = e^{(j+j')\pi i/12}\,(\tau)^{1/2}\,\chi(G)\,\psi_{-j'}(\tau)\ , \tag{A.8}$$

where $jj'=1 \bmod N$ and the character $\chi(G)$ has to be calculated on a case by case basis (see chapter 2 of [80] for details).

The transformations of the eta related functions show us that the functions $f^k(\tau)$ and its square root can transform with non-trivial character. In particular, one can show that for $N=7$, $f^{(1)}(\tau)$ and for $N=3$, $\sqrt{f^{(4)}(\tau)}$ transform with character. As these two functions enter the weak Jacobi forms that are used to construct the Siegel modular forms $\Phi_1(\mathbf{Z})$ and $\Delta_2(\mathbf{Z})$ respectively, these two Siegel modular forms will transform with non-trivial character[83]. This is the basis for our claim that $\Delta_2(\mathbf{Z})$ must transform with non-trivial character and is consistent with the observation of Jatkar-Sen regarding $\Phi_1(\mathbf{Z})$.

B

Eisenstein Series at level N

B.1 Prime N

Let $E_2^*(\tau)$ denote the weight two non-holomorphic modular form of $SL(2,\mathbb{Z})$. It is given by

$$E_2^*(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\ q^n\ + \frac{3}{\pi\ \mathrm{Im}\tau}\ , \tag{B.1}$$

where $\sigma_\ell(n) = \sum_{1\leq d|n} d^\ell$. The combination

$$E_N(\tau) = \frac{1}{N-1}\Big(NE_2^*(N\tau) - E_2^*(\tau)\Big) = \tfrac{12i}{\pi(N-1)}\partial_\tau\big[\ln\eta(\tau) - \ln\eta(N\tau)\big] \tag{B.2}$$

is a weight two holomorphic modular form of $\Gamma_0(N)$ with constant coefficient equal to 1[122, Theorem 5.8]. Note the cancellation of the non-holomorphic pieces. Thus, at level $N > 1$, the Eisenstein series produces a weight two modular form. For example[1],

$$E_2(\tau) = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + 96q^6 + \cdots \tag{B.3}$$

is the weight-two Eisenstein series at level 2. At levels 3 and 5, one has

$$\begin{aligned} E_3(\tau) &= 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + 36q^6 + \cdots \\ E_5(\tau) &= 1 + 6q + 18q^2 + 24q^3 + 42q^4 + 6q^5 + 72q^6 + \cdots \end{aligned} \tag{B.4}$$

[1] All expansions for the Eisenstein series given here have been obtained using the mathematics software SAGE[123]. We are grateful to the authors of SAGE for making their software freely available. It was easy for us to verify Eq. (B.10) using SAGE to the desired order.

B.2 Composite N

Suppose $M|N$, then one has $\Gamma_0(N) \subset \Gamma_0(M)$. Thus, for composite N, the Eisenstein series at level M is also a modular form at level N. For instance at level four, one has two Eisenstein series: $E_2(\tau)$ and

$$E_4(\tau) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + \cdots \tag{B.5}$$

At level six, one has three Eisenstein series: $E_2(\tau)$, $E_3(\tau)$ and

$$\widehat{E}_6(\tau) = 5/24 + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \cdots \tag{B.6}$$

At level eight, one has three Eisenstein series: $E_2(\tau)$, $E_4(\tau)$ and

$$\widehat{E}_8(\tau) = 7/24 + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \cdots \tag{B.7}$$

$\widehat{E}_N(\tau)$ refer to Eisenstein series normalized such that the coefficient of q is $+1$. It is known that all Eisenstein series in this normalization have integral coefficients except for the constant term[122].

B.3 Fourier transform about the cusp at 0

The modular transformation, S, under which $\tau \to -1/\tau$ maps the cusp at 0 to the cusp at $i\infty$. When N is prime, $\Gamma_0(N)$ has only these two cusps. One may wish to obtain the Fourier expansion about the cusp at 0 – this is done by mapping 0 to $i\infty$ using the S transform. To obtain the transform of the Eisenstein series, first consider

$$\begin{aligned} E_2^*(N\tau)\big|_S &= (\tau)^{-2}\ E_2^*(NS\cdot\tau) \\ &= (\tau)^{-2}\ E_2^*(-N/\tau) = (\tau)^{-2}(\tau/N)^2 E_2^*(\tau/N) = \frac{1}{N^2}\ E_2^*\left(\tfrac{\tau}{N}\right)\ . \end{aligned} \tag{B.8}$$

Using this result, it is easy to see that[2]

$$E_N(\tau)\big|_S = -\frac{1}{N} E_N\left(\tfrac{\tau}{N}\right)\ . \tag{B.9}$$

[2] We caution the reader that the subscript N denotes the level and *not* the weight of the Eisenstein series. All Eisenstein series considered in this appendix are of weight two.

Note that $\tau = 0$ in the LHS corresponds to $\tau = i\infty$ in the RHS of the above equation. Thus, given the Fourier expansion at $i\infty$, we can obtain the Fourier expansion about 0. Notice the appearance of fractional powers of q, $q^{1/N}$ to be precise, at this cusp. This is expected as the width of the cusp at 0 is N. Also, note that the above formula is valid for all N, not necessarily prime.

Another useful addition formula for the Eisenstein series is the following:

$$E_4(\tau) + E_4(\tau + \tfrac{1}{2}) = 2\ E_2(2\tau)\ . \tag{B.10}$$

This formula was experimentally obtained by us and its veracity has been checked to around twenty orders in the Fourier expansion.

B.4 Fourier transform about other cusps

The same method can be used to obtain the expansion about other cusps. Again we will need to map the cusp to $i\infty$ and then track the transformation of the non-holomorphic Eisenstein series. Let us do a specific example that is of interest in this paper. Let $N = 4$ and consider the cusp at 1/2. $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$ maps 1/2 to $i\infty$.

$$\begin{aligned} E_4(\tau)\big|_{ST^2S} = -\frac{1}{4}E_4(\tfrac{\tau}{4})\big|_{ST^2} = -\frac{1}{4}E_4(\tfrac{\tau}{4} + \tfrac{1}{2})\big|_S \\ = -\frac{1}{4}\left(2E_2(\tfrac{\tau}{2})\big|_S - E_4(\tfrac{\tau}{4})\big|_S\right) = \left(E_2(\tau) - E_4(\tau)\right) \end{aligned} \tag{B.11}$$

In the penultimate step, we made use of Eq. (B.10) in order to write $E_4(\frac{\tau}{4} + \frac{1}{2})$ in terms of objects with known S-transformations. The final answer is in terms of Eisenstein series whose Fourier coefficients are known thus giving us the expansion of $E_4(\tau)$ about the cusp at 1/2.

For the CHL models with $N = 6$ and $N = 8$, it appears that there are no standard methods to determined the Fourier expansion of $E_6(\tau)$ and $E_8(\tau)$ about all the cusps – this is a minor technical hurdle that needs to be surmounted to complete the computation of the twisted elliptic genus in the corresponding CHL models. It would be helpful if one can obtain identities similar to the one given in Eq. (B.10).

C

Explicit formulae for $\Delta_{k/2}(\mathbf{Z})$

We note that $\Delta_{k/2}(\mathbf{Z})$ is symmetric under the exchange $z_1 \leftrightarrow z_3$ and is anti-symmetric under $z_2 \to -z_2$ for all values of k.

$$
\begin{aligned}
\Delta_5 \;=\;& \left(-\frac{1}{\sqrt{r}} + \sqrt{r}\right)\sqrt{q}\sqrt{s} + \left(\frac{9}{r^{\frac{5}{2}}} - \frac{93}{r^{\frac{3}{2}}} + \frac{90}{\sqrt{r}} - 90\sqrt{r} + 93\, r^{\frac{3}{2}} - 9\, r^{\frac{5}{2}}\right) q^{\frac{3}{2}} s^{\frac{3}{2}} \\
+\;& \left(r^{-\frac{3}{2}} + \frac{9}{\sqrt{r}} - 9\sqrt{r} - r^{\frac{3}{2}}\right)\left(q^{\frac{3}{2}}\sqrt{s} + \sqrt{q}\, s^{\frac{3}{2}}\right) \\
+\;& \left(\frac{-9}{r^{\frac{3}{2}}} - \frac{27}{\sqrt{r}} + 27\sqrt{r} + 9\, r^{\frac{3}{2}}\right)\left(q^{\frac{5}{2}}\sqrt{s} + \sqrt{q}\, s^{\frac{5}{2}}\right) \\
+\;& \left(-r^{-\frac{5}{2}} + \frac{27}{r^{\frac{3}{2}}} + \frac{12}{\sqrt{r}} - 12\sqrt{r} - 27\, r^{\frac{3}{2}} + r^{\frac{5}{2}}\right)\left(q^{\frac{7}{2}}\sqrt{s} + \sqrt{q}\, s^{\frac{7}{2}}\right) \\
+\;& \left(\frac{9}{r^{\frac{5}{2}}} - \frac{12}{r^{\frac{3}{2}}} + \frac{90}{\sqrt{r}} - 90\sqrt{r} + 12\, r^{\frac{3}{2}} - 9\, r^{\frac{5}{2}}\right)\left(q^{\frac{9}{2}}\sqrt{s} + \sqrt{q}\, s^{\frac{9}{2}}\right) \\
+\;& \left(\frac{-27}{r^{\frac{5}{2}}} - \frac{90}{r^{\frac{3}{2}}} - \frac{135}{\sqrt{r}} + 135\sqrt{r} + 90\, r^{\frac{3}{2}} + 27\, r^{\frac{5}{2}}\right)\left(q^{\frac{11}{2}}\sqrt{s} + \sqrt{q}\, s^{\frac{11}{2}}\right) \\
+\;& \left(r^{-\frac{7}{2}} + \frac{12}{r^{\frac{5}{2}}} + \frac{135}{r^{\frac{3}{2}}} - \frac{54}{\sqrt{r}} + 54\sqrt{r} - 135\, r^{\frac{3}{2}} - 12\, r^{\frac{5}{2}} - r^{\frac{7}{2}}\right)\left(q^{\frac{13}{2}}\sqrt{s} + \sqrt{q}\, s^{\frac{13}{2}}\right) + \ldots
\end{aligned}
$$

$$\begin{aligned}
\Delta_3 \;=\;& \left(-\frac{1}{\sqrt{r}}+\sqrt{r}\right)\sqrt{q}\sqrt{s}+\left(r^{-\frac{5}{2}}-\frac{5}{r^{\frac{3}{2}}}-\frac{6}{\sqrt{r}}+6\sqrt{r}+5\,r^{\frac{3}{2}}-r^{\frac{5}{2}}\right)q^{\frac{3}{2}}s^{\frac{3}{2}}\\
+\;& \left(r^{-\frac{3}{2}}+\frac{1}{\sqrt{r}}-\sqrt{r}-r^{\frac{3}{2}}\right)\left(q^{\frac{3}{2}}\sqrt{s}+\sqrt{q}\,s^{\frac{3}{2}}\right)\\
+\;& \left(-r^{-\frac{3}{2}}+\frac{5}{\sqrt{r}}-5\sqrt{r}+r^{\frac{3}{2}}\right)\left(q^{\frac{5}{2}}\sqrt{s}+\sqrt{q}\,s^{\frac{5}{2}}\right)\\
+\;& \left(-r^{-\frac{5}{2}}-\frac{5}{r^{\frac{3}{2}}}-\frac{4}{\sqrt{r}}+4\sqrt{r}+5\,r^{\frac{3}{2}}+r^{\frac{5}{2}}\right)\left(q^{\frac{7}{2}}\sqrt{s}+\sqrt{q}\,s^{\frac{7}{2}}\right)\\
+\;& \left(r^{-\frac{5}{2}}+\frac{4}{r^{\frac{3}{2}}}-\frac{6}{\sqrt{r}}+6\sqrt{r}-4\,r^{\frac{3}{2}}-r^{\frac{5}{2}}\right)\left(q^{\frac{9}{2}}\sqrt{s}+\sqrt{q}\,s^{\frac{9}{2}}\right)\\
+\;& \left(\frac{5}{r^{\frac{5}{2}}}+\frac{6}{r^{\frac{3}{2}}}+\frac{1}{\sqrt{r}}-\sqrt{r}-6\,r^{\frac{3}{2}}-5\,r^{\frac{5}{2}}\right)\left(q^{\frac{11}{2}}\sqrt{s}+\sqrt{q}\,s^{\frac{11}{2}}\right)\\
+\;& \left(r^{-\frac{7}{2}}-\frac{4}{r^{\frac{5}{2}}}-r^{-\left(\frac{3}{2}\right)}-\frac{6}{\sqrt{r}}+6\sqrt{r}+r^{\frac{3}{2}}+4\,r^{\frac{5}{2}}-r^{\frac{7}{2}}\right)\left(q^{\frac{13}{2}}\sqrt{s}+\sqrt{q}\,s^{\frac{13}{2}}\right)+\dots
\end{aligned}$$

$$\begin{aligned}
\Delta_2 \;=\;& \left(\sqrt{r}-\frac{1}{\sqrt{r}}\right)\sqrt{q}\sqrt{s}+\left(3r^{3/2}-\frac{3}{r^{3/2}}\right)q^{3/2}s^{3/2}+\left(\frac{1}{r^{3/2}}-r^{3/2}\right)\left(\sqrt{s}q^{3/2}+s^{3/2}\sqrt{q}\right)\\
+\;& \left(r^{5/2}-3\sqrt{r}+\frac{3}{\sqrt{r}}-\frac{1}{r^{5/2}}\right)\left(\sqrt{s}q^{7/2}+s^{7/2}\sqrt{q}\right)+\left(3r^{3/2}-\frac{3}{r^{3/2}}\right)\left(\sqrt{s}q^{9/2}+s^{9/2}\sqrt{q}\right)\\
+\;& \left(-r^{7/2}-3r^{5/2}+\frac{3}{r^{5/2}}+\frac{1}{r^{7/2}}\right)\left(\sqrt{s}q^{13/2}+s^{13/2}\sqrt{q}\right)\\
+\;& \left(3r^{7/2}+5\sqrt{r}-\frac{5}{\sqrt{r}}-\frac{3}{r^{7/2}}\right)\left(\sqrt{s}q^{19/2}+s^{19/2}\sqrt{q}\right)+\dots
\end{aligned}$$

$$\begin{aligned}
\Delta_1 &= \left(\sqrt{r}-\frac{1}{\sqrt{r}}\right)\sqrt{q}\sqrt{s}+\left(-r^{3/2}+\sqrt{r}-\frac{1}{\sqrt{r}}+\frac{1}{r^{3/2}}\right)\left(\sqrt{s}q^{3/2}+s^{3/2}\sqrt{q}\right)\\
&+ \left(r^{5/2}-2r^{3/2}+5\sqrt{r}-\frac{5}{\sqrt{r}}+\frac{2}{r^{3/2}}-\frac{1}{r^{5/2}}\right)q^{3/2}s^{3/2}\\
&+ \left(-r^{3/2}+2\sqrt{r}-\frac{2}{\sqrt{r}}+\frac{1}{r^{3/2}}\right)\left(\sqrt{s}q^{5/2}+s^{5/2}\sqrt{q}\right)\\
&+ \left(r^{5/2}-2r^{3/2}+3\sqrt{r}-\frac{3}{\sqrt{r}}+\frac{2}{r^{3/2}}-\frac{1}{r^{5/2}}\right)\left(\sqrt{s}q^{7/2}+s^{7/2}\sqrt{q}\right)\\
&+ \left(r^{5/2}-3r^{3/2}+5\sqrt{r}-\frac{5}{\sqrt{r}}+\frac{3}{r^{3/2}}-\frac{1}{r^{5/2}}\right)\left(\sqrt{s}q^{9/2}+s^{9/2}\sqrt{q}\right)\\
&+ \left(2r^{5/2}-5r^{3/2}+5\sqrt{r}-\frac{5}{\sqrt{r}}+\frac{5}{r^{3/2}}-\frac{2}{r^{5/2}}\right)\left(\sqrt{s}q^{11/2}+s^{11/2}\sqrt{q}\right)\\
&+ \left(-r^{7/2}+3r^{5/2}-5r^{3/2}+9\sqrt{r}-\frac{9}{\sqrt{r}}+\frac{5}{r^{3/2}}-\frac{3}{r^{5/2}}+\frac{1}{r^{7/2}}\right)\left(\sqrt{s}q^{13/2}+s^{13/2}\sqrt{q}\right)+\ldots
\end{aligned}$$

$$\begin{aligned}
\Delta_{3/2} &= \left(\sqrt{r}-\frac{1}{\sqrt{r}}\right)\sqrt{s}\sqrt{q}+\left(r^{5/2}-r^{3/2}+2\sqrt{r}-\frac{2}{\sqrt{r}}+\frac{1}{r^{3/2}}-\frac{1}{r^{5/2}}\right)s^{3/2}q^{3/2}\\
&+\left(-r^{3/2}+\sqrt{r}-\frac{1}{\sqrt{r}}+\frac{1}{r^{3/2}}\right)\left(\sqrt{q}s^{3/2}+\sqrt{s}q^{3/2}\right)\\
&+\left(-r^{3/2}+\sqrt{r}-\frac{1}{\sqrt{r}}+\frac{1}{r^{3/2}}\right)\left(\sqrt{q}s^{5/2}+\sqrt{s}q^{5/2}\right)\\
&+\left(r^{5/2}-r^{3/2}+2\sqrt{r}-\frac{2}{\sqrt{r}}+\frac{1}{r^{3/2}}-\frac{1}{r^{5/2}}\right)\left(\sqrt{q}s^{7/2}+\sqrt{s}q^{7/2}\right)\\
&+\left(r^{5/2}-2r^{3/2}+\frac{2}{r^{3/2}}-\frac{1}{r^{5/2}}\right)\left(\sqrt{q}s^{9/2}+\sqrt{s}q^{9/2}\right)\\
&+\left(r^{5/2}+\sqrt{r}-\frac{1}{\sqrt{r}}-\frac{1}{r^{5/2}}\right)\left(\sqrt{q}s^{11/2}+\sqrt{s}q^{11/2}\right)\\
&+\left(-r^{7/2}+2r^{5/2}-r^{3/2}+2\sqrt{r}-\frac{2}{\sqrt{r}}+\frac{1}{r^{3/2}}-\frac{2}{r^{5/2}}+\frac{1}{r^{7/2}}\right)\left(\sqrt{q}s^{13/2}+\sqrt{s}q^{13/2}\right)
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\Delta}_{3/2} \;=\;& \left(-\frac{1}{\sqrt{r}}+\sqrt{r}\right)\sqrt{s}\sqrt{q_h}+\left(2r^{3/2}-2\sqrt{r}+\frac{2}{\sqrt{r}}-\frac{2}{r^{3/2}}\right)s^{3/2}{q_h}^{3/2}\\
&+\left(\frac{2}{\sqrt{r}}-2\sqrt{r}\right)\left(\sqrt{q_h}s^{\frac{3}{2}}+\sqrt{s}{q_h}^{\frac{3}{2}}\right)+\left(\frac{2}{\sqrt{r}}-2\sqrt{r}\right)\left(\sqrt{q_h}s^{\frac{5}{2}}+\sqrt{s}{q_h}^{\frac{5}{2}}\right)\\
&+\left(\frac{-4}{\sqrt{r}}+4\sqrt{r}\right)\left(\sqrt{q_h}s^{\frac{7}{2}}+\sqrt{s}{q_h}^{\frac{7}{2}}\right)\\
&+\left(r^{-\frac{3}{2}}-\frac{2}{\sqrt{r}}+2\sqrt{r}-r^{\frac{3}{2}}\right)\left(\sqrt{q_h}s^{\frac{9}{2}}+\sqrt{s}{q_h}^{\frac{9}{2}}\right)\\
&+\left(\frac{-2}{r^{\frac{3}{2}}}+2\,r^{\frac{3}{2}}\right)\left(\sqrt{q_h}s^{\frac{11}{2}}+\sqrt{s}{q_h}^{\frac{11}{2}}\right)\\
&+\left(\frac{-2}{r^{\frac{3}{2}}}+\frac{4}{\sqrt{r}}-4\sqrt{r}+2\,r^{\frac{3}{2}}\right)\left(\sqrt{q_h}s^{\frac{13}{2}}+\sqrt{s}{q_h}^{\frac{13}{2}}\right)+\ldots\ ,
\end{aligned}
$$

where $q_h \equiv q^{1/4}$. The expression is symmetric under the exchange $q \leftrightarrow s^4$ and antisymmetric under $r \to r^{-1}$. An all-orders proof follows from the properties of the even genus-two theta constants.

D

The Jacobi and the genus-two modular groups

The group $Sp(2,\mathbb{Z})$ is the set of 4×4 matrices written in terms of four 2×2 matrices A, B, C, D as[1]

$$M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$$

satisfying $AB^T = BA^T$, $CD^T = DC^T$ and $AD^T - BC^T = I$. The congruence subgroup $\widehat{G}_0(N)$ of $Sp(2,\mathbb{Z})$ is given by the set of matrices such that $C = 0 \bmod N$. This group acts naturally on the Siegel upper half space, $\mathbb{H}_2$, as

$$\mathbf{Z}=\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3\end{pmatrix} \longmapsto M\cdot\mathbf{Z} \equiv (A\mathbf{Z}+B)(C\mathbf{Z}+D)^{-1}\ . \tag{D.1}$$

The Jacobi group $\Gamma^J = SL(2,\mathbb{Z}) \ltimes H(\mathbb{Z})$ is the sub-group of $Sp(2,\mathbb{Z})$ that preserves the one-dimensional cusp $z_3 = i\infty$. The $SL(2,\mathbb{Z})$ is generated by the embedding of $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ in $Sp(2,\mathbb{Z})$

$$g_1(a,b;c,d) \equiv \begin{pmatrix} a & 0 & b & 0\\ 0 & 1 & 0 & 0\\ c & 0 & d & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\ . \tag{D.2}$$

The above matrix acts on $\mathbb{H}_2$ as

$$(z_1,z_2,z_3) \longrightarrow \left(\frac{az_1+b}{cz_1+d},\ \frac{z_2}{cz_1+d},\ z_3 - \frac{cz_2^2}{cz_1+d}\right)\ , \tag{D.3}$$

[1]This section is based on the book by Eichler and Zagier[83].

with $\det(C\mathbf{Z}+D) = (cz_1+d)$. The Heisenberg group, $H(\mathbb{Z})$, is generated by $Sp(2,\mathbb{Z})$ matrices of the form

$$g_2(\lambda,\mu,\kappa) \equiv \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with } \lambda,\mu,\kappa \in \mathbb{Z} \tag{D.4}$$

The above matrix acts on $\mathbb{H}_2$ as

$$(z_1, z_2, z_3) \longrightarrow \big(z_1,\ \lambda z_1 + z_2 + \mu,\ z_3 + \lambda^2 z_1 + 2\lambda z_2 + \lambda\mu\big)\ , \tag{D.5}$$

with $\det(C\mathbf{Z}+D) = 1$. It is easy to see that Γ^J preserves the one-dimensional cusp at $\mathrm{Im}(z_3) = \infty$.

The full group $Sp(2,\mathbb{Z})$ is generated by adding the exchange element to the group Γ^J.

$$g_3 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\ . \tag{D.6}$$

This acts on $\mathbb{H}_2$ exchanging $z_1 \leftrightarrow z_3$. The subgroup $\widehat{G}_0(N)$ is generated by considering the same three sets of matrices with the additional condition that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ i.e., $c = 0 \bmod N$ in Eq. (D.2). Further, we will call the corresponding Jacobi group $\Gamma_0(N)^J$.

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