

**INTRODUCTION TO MANIFOLDS: SIMPLE TO COMPLEX**  
**WITH SOME NUMERICAL COMPUTATIONS**

**A PROJECT REPORT**

*submitted in partial fulfilment of the requirements  
for the award of the degree of*

**MASTER OF SCIENCE**  
**in**  
**PHYSICS**

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# CERTIFICATE

The project titled **Introduction to Manifolds: Simple to Complex (with some numerical computations)**, was completed by **Mr. Sidharth Kshatriya** under my guidance during the academic year 2006-2007. I certify that this is an original project report resulting from the work completed during this period.

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**Sidharth Kshatriya**

# ABSTRACT

This project report aims to introduce Manifolds. Manifolds are fundamental structures in Differential Geometry. The study of Manifolds is useful in various branches of Theoretical Physics, especially High Energy Physics and General Relativity. For instance, Einstein's theory of General Relativity conjectures that space-time forms a 4 dimensional pseudo-Riemannian Manifold. Superstring Theory explains the compactification of extra dimensions by using Calabi-Yau Manifolds.

Manifolds are abstract mathematical spaces that look locally like  $\mathbb{R}^n$  but may have a more complicated large scale structure. The surface of Earth is a simple example: At small distances it looks like the Euclidean  $\mathbb{R}^2$  but from far away it is  $\mathbb{S}^2$ , the two dimensional surface of a sphere. The behaviour at the small scale and large scale can be totally different. For instance, in  $\mathbb{R}^2$  parallel lines never meet while all lines eventually meet in  $\mathbb{S}^2$ . Because all Manifolds are locally like Euclidean Space we can develop common mathematical techniques to study extremely different kinds of spaces.

We can define increasingly complicated structures on Manifolds so that we may do Calculus on them or define concepts of distance and angles on them. We may also want to study Manifolds in terms of complex variables and perform Complex Calculus on them. In this report we look at the whole hierarchy of Manifolds. We start from Simple Manifolds and progress to Differentiable Manifolds, Riemannian Manifolds and lastly Complex Manifolds. Within Complex Manifolds we study Hermetian Manifolds and Kähler Manifolds. Orbifolds, another special kind of Manifold, are also introduced. All the related mathematics and concepts such as Vector Fields, Tangent Spaces, Metrics, Curvature, Parallel Transport and Connection are explained.

Subsequently, numerical calculations are performed on the  $\mathbb{C}^3/\mathbb{Z}_3$  Orbifold.  $\mathbb{C}^3/\mathbb{Z}_3$  is a Kähler Manifold, important in String Theory. Given a potential function we calculate Ricci-flat (and non-flat) metrics and the Ricci curvature for this Orbifold.

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**NOTE: All figures have been made by me**

# CHAPTER 1

## SIMPLE & DIFFERENTIAL MANIFOLDS

### 1.1 What is a Manifold?

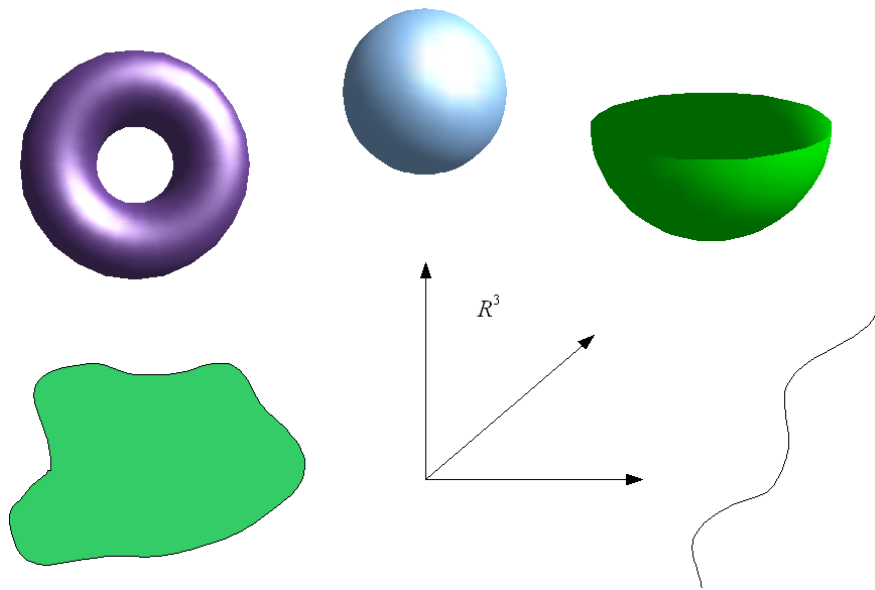


Figure 1.1: What is a Manifold?

A few examples of Manifolds are illustrated above. A Torus,  $\mathbb{R}^3$ , a sphere, an arbitrary curve or surface (in general an arbitrary  $n$  dimensional object), the surface of a human being etc. are all valid examples of a Manifold. A Manifold can be *compact* (have boundaries) like a sphere or a cone or it can be infinite like  $\mathbb{R}^3, \mathbb{R}^n, \mathbb{C}^n$ .

Given that Manifolds can come in such a wide variety of different shapes and sizes how can we study all of them together? The answer is simple. At a high enough “magnification” or in mathematical terminology *locally each* manifold resembles  $\mathbb{R}^n$ <sup>1</sup>. The surface of Earth is a simple example: At small distances it looks like the Euclidean  $\mathbb{R}^2$  but from far away it is  $\mathbb{S}^2$ , the two dimensional surface of a sphere. This property allows us to depict parts of the Earth’s surface in maps which are  $\mathbb{R}^2$ . But this fails if we want to show large parts of the Earth on the map and we get all kinds of distortions in showing the Earth in the  $\mathbb{R}^2$  representation.

Note that the behaviour locally and at large scale can be totally different. For instance, locally  $\mathbb{S}^2$  is like  $\mathbb{R}^2$  where parallel lines never meet while all lines eventually meet in  $\mathbb{S}^2$ .

<sup>1</sup>If its a complex space it will resemble  $\mathbb{C}^n$



## 1.2 Simple Manifolds

So now that we know what a Manifold is, we would like to assign coordinates to the points on it. In the case of the *surface* of a sphere of radius 1 we could say that each point has a coordinate  $(x, y, z)$  where  $x^2 + y^2 + z^2 = 1$ . This is not very elegant because its a 2 dimensional surface and we are using 3 coordinates to represent it. One should be able to use only two coordinates. So lets do that. We represent a point  $P$  on the surface as  $(\theta, \phi)$  where  $\theta$  is the polar angle and  $\phi$  is the azimuthal angle.  $\theta \in [0, \pi)$  and  $\phi \in [0, 2\pi)$ .

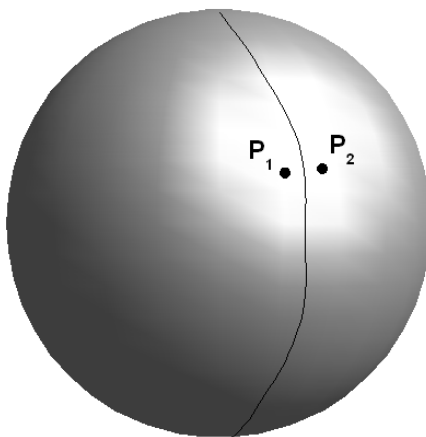


Figure 1.2: As we travel across the line from  $P_1$  to  $P_2$ ,  $\phi$  undergoes a jump from  $2\pi$  to 0

However the  $(\theta, \phi)$  representation has its own problems. For instance, lets travel from point  $P_1$  to  $P_2$  (see figure) . As we do that, the  $\phi$  coordinate “jumps” from  $2\pi$  to 0 when we cross the line  $\phi = 0$  line <sup>2</sup>.

This “jump” is not desirable and makes this coordinate system less than ideal (these “jumps” may cause problems in differentiation, taking limits etc). This “jump” also occurs with  $\theta$  when it goes beyond  $\pi$ . We can solve this “jump” problem by allowing  $\phi$  to increase beyond  $2\pi$  and  $\theta$  to increase beyond  $\pi$ . So each point  $(\theta, \phi) \equiv (\theta + \pi n, \phi + 2\pi m)$  where  $m, n \in \mathbb{Z}$ . In other words, each point can be represented by multiple coordinates. Again this is not a satisfactory solution.

Another problem that we notice is that  $\phi$  is undefined when  $\theta = \pi$  or  $\theta = 0$ .

Is is possible to find a coordinate system in which each point has a **unique** coordinate and there are no “jumps” in the coordinate values for **nearby** points? The answer is YES! (But with a few complications)

---

<sup>2</sup>strictly speaking, since  $\phi < 2\pi$  the jump is from some  $2\pi - \epsilon$  to 0 where  $\epsilon$  is arbitrarily small

## 1.2.1 Motivating Manifolds through a Canonical Example

### 1.2.1.1 North Pole Stereographic Projection

Let us consider a different kind of coordinate system, the North Pole Projection (NPP) system. In this system we draw a line from North Pole through any point P on the surface of the sphere (Note that radius= 1 in our examples). We continue the line until it crosses the plane that cuts the sphere at  $\theta = \pi/2$  (See figure). The point of crossing the plane is represented by the coordinate  $(X, Y)$

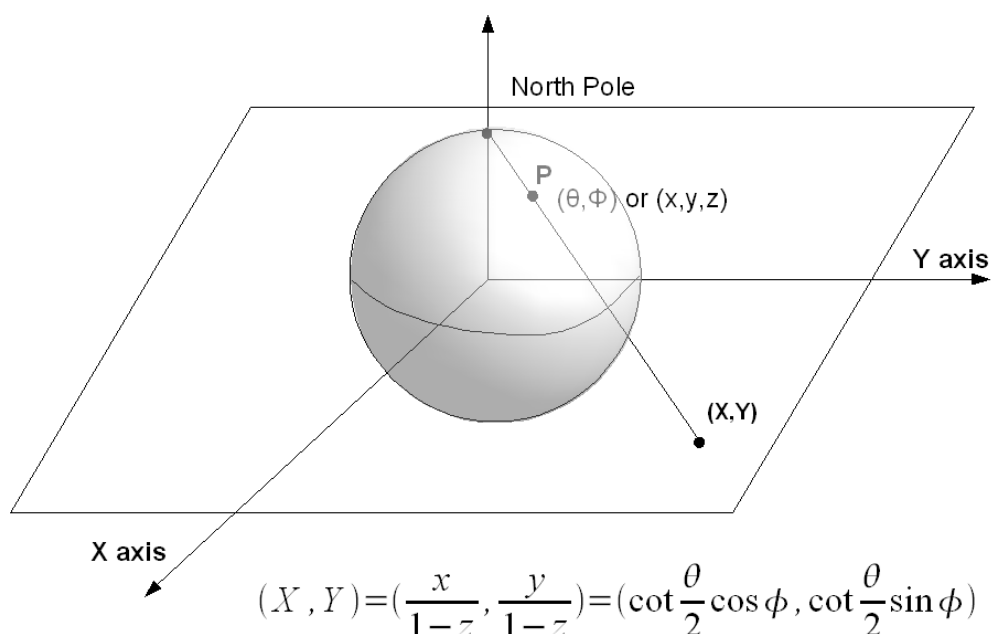


Figure 1.3: North Pole Projection Stereographic Coordinates

We can show (very easily) that:

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \left(\cot \frac{\theta}{2} \cos \phi, \cot \frac{\theta}{2} \sin \phi\right)$$

Note that we have achieved the following objectives:

- each point has a unique coordinate
- there are no “jumps” in the coordinate values for nearby points

However there is additional complication: **we cannot represent the North Pole point in our coordinate system.** The line joining the North Pole to the North Pole is parallel to the plane. Since it never touches it, we don't have a coordinate for the North Pole. So to represent the North Pole we develop the South Pole Stereographic Projection.

### 1.2.1.2 South Pole Stereographic Projection

Similar to above, let us take a line from the South Pole through the point P and see where it cuts the  $\theta = \pi/2$  plane. Let the point be  $(U, V)$  (See figure). We call the coordinate the South Pole Projection (SPP) coordinate.

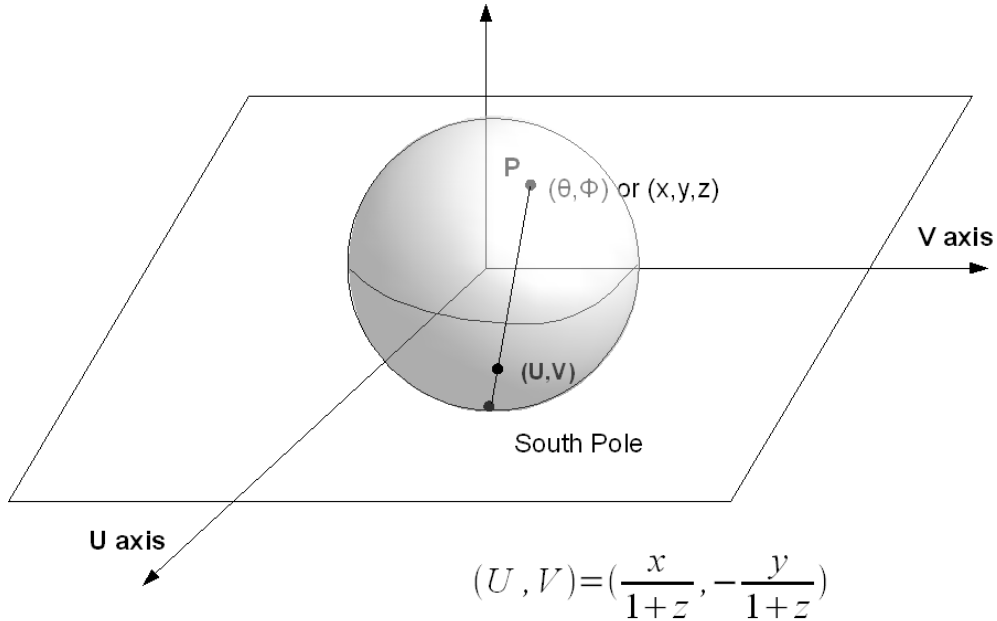


Figure 1.4: South Pole Projection Stereographic Coordinates

This coordinate system gives the North Pole coordinate as  $(U, V) = (0, 0)$ . We *can* represent the North Pole in it but we *cannot* now represent the South Pole in it. The line from the South Pole to the South Pole never touches the coordinate plane.

Essentially we have achieved the following. We have two *local* coordinate systems. They are called local because they do not represent all the points in the Manifold. The SPP system gives coordinates for points in

$$\mathbb{S}^2 - \{\text{South Pole}\}$$

and the NPP system gives coordinates for points in

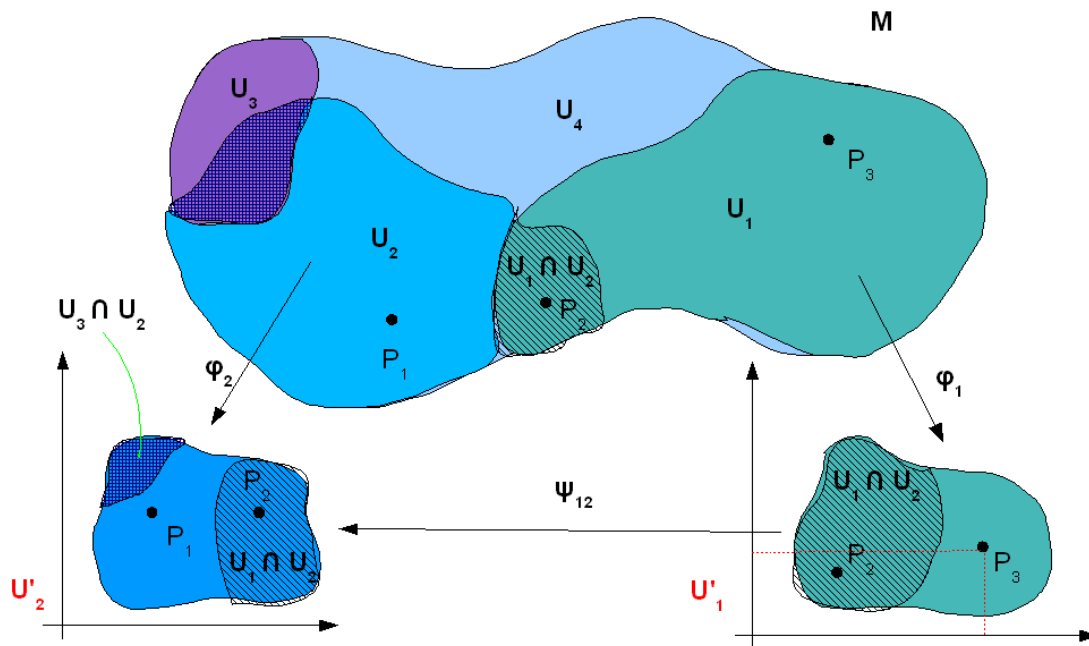
$$\mathbb{S}^2 - \{\text{North Pole}\}$$

Using multiple local coordinate systems (in our example two) we can always cover an arbitrary Manifold such that each point has a coordinate in *at least* one coordinate system.

However there is an important consistency criterion for the points that have coordinates in multiple coordinate functions. We require that for such points the transition function for

coordinates from one local system to another be *smooth*. In our example, the mathematical function that takes us from  $(U, V)$  to  $(X, Y)$  (or vice versa) must be smooth. If the transition function is infinitely differentiable then the Manifold is a **Differentiable Manifold**.

### 1.2.2 Some Formalism: Charts & Atlases



We now develop some formalism. The formalism and notation is similar to [1, pg. 133] but we motivate it with our own illustration. In the above diagram we have “patched” some Manifold  $M$  having dimension  $m$  with open sets  $U_i$  such that  $\bigcup U_i = M$  where  $i = 1 \dots 4$ . Each point should be present in at least one  $U_i$ . Regions with overlaps are shown in hatched lines. These regions are also marked  $U_1 \cap U_2$  etc. We may even have regions like  $U_1 \cap U_2 \cap U_3$  for some other Manifold but for simplicity we assume that is not the case here. Of course, the nature of the overlaps between  $U_i$  and the number of  $U_i$  required to cover the whole Manifold depends on the Manifold itself.

Each  $U_i$  is associated with a map  $\varphi_i$  that takes a point and gives its coordinates in the local coordinate system  $U'_i$ . Given a point  $P \in U_i$ ,  $\varphi_i(P) = (x_i^1, x_i^2, \dots, x_i^m)$ . The superscripts here are not powers but just labels to distinguish between the coordinate components. It is important to note that points exist in a Manifold regardless of what coordinates they have. Points have an existence independent from their representation in some local coordinate system.

Lets look at some representative points on the diagram. Take  $P_3 \in U_1$ .  $\varphi_1(P_3)$

has some coordinate in the local coordinate system  $U'_1$  on the lower right hand side (e.g. (23,3.5) if  $m = 2$ <sup>3</sup>). Take  $P_2 \in U_1 \cap U_2$ . We have two coordinates representations  $\varphi_1(P_2)$  and  $\varphi_2(P_2)$ .

Let  $\psi_{12}$  be a map that takes us from  $U'_1$  to  $U'_2$ . Restated,  $\psi_{12}(x^\mu) \equiv \varphi_2\varphi_1^{-1}$ . We can understand this as the following: Let  $x^\mu$  where  $(1 \leq \mu \leq m)$  be the local coordinate (in system  $U'_1$ ) of a point  $A \in U_1 \cap U_2$ . The inverse map  $\varphi_1^{-1}$  acts on a local coordinate  $x^\mu$  in  $U'_1$  and gives us  $A$ . Then the map  $\varphi_2$  acts on point  $A$  and gives us the local coordinates  $y^\nu$  ( $1 \leq \nu \leq m$ ) in the system  $U'_2$ . In general  $\psi_{ij}(x^\mu) \equiv \varphi_j\varphi_i^{-1}$ .

In function notation we can represent  $\psi_{12}$  as  $m$  functions

$$\begin{aligned} x^1(y^1, y^2, \dots, y^m) \\ x^2(y^1, y^2, \dots, y^m) \\ x^3(y^1, y^2, \dots, y^m) \\ \vdots \\ x^m(y^1, y^2, \dots, y^m) \end{aligned}$$

These  $m$  functions are called transition functions because they allow us to go from the local coordinate system  $(x^1, x^2, \dots, x^m) \in U'_1$  to another local coordinate system  $(y^1, y^2, \dots, y^m) \in U'_2$ . For the Manifold to be a **Differentiable Manifold** each of these functions must be infinitely differentiable or  $\mathbb{C}^\infty$ .

In general, if  $z_i^\mu$  is the coordinate in the  $i$ th local system then all the transition functions  $z_i^\mu(z_j^1, z_j^2, z_j^3, \dots, z_j^m)$  should be  $\mathbb{C}^\infty$  where  $1 \leq i, j, \mu \leq m$ . Note that the transition functions only make sense on points that are present in the originating “patch” and the destination “patch”.

### 1.2.3 The Three Rules of Manifolds

We summarize our introduction to Manifolds with three rules (from [1, pg. 132]):

- Nearby Points have nearby coordinates in at least one coordinate system
- Every point has unique coordinates in each coordinate system that contains it
- If two coordinate systems overlap, then in the region of the overlap, local coordinates of the point are related to each other in a sufficiently smooth way

---

<sup>3</sup>we are using cartesian coordinates in the  $U'_1$  local coordinate system. This is just an example. We may use polar or any other coordinate system

### 1.3 Some Basic Concepts

We introduce Tangent Spaces and Vector Fields. These concepts are fundamental to understanding Manifolds.

#### 1.3.1 Tangent Spaces

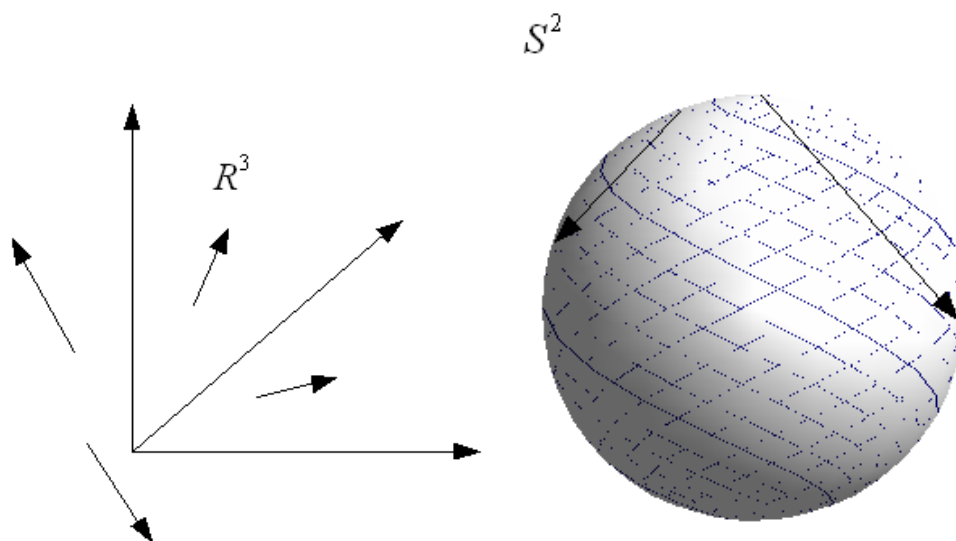


Figure 1.5: Vectors can't just start and end anywhere like in  $\mathbb{R}^n$

In  $\mathbb{R}^3$ , vectors can start and end anywhere. In the above diagram we have shown that. However in other Manifolds like  $S^2$  this is not true. It does **not** make sense for a vector to start and end anywhere. In the above diagram we have shown two invalid examples for  $S^2$  of vectors that start on the surface and then cut through the Manifold onto another point.  $S^2$  is a two dimensional Manifold. The body of the vectors is, in a way, traveling outside the Manifold. What is the solution?

The solution is to require each vector to live in a Tangent Space. As the name implies, the Tangent Space is tangent to the Manifold. Each point on the Manifold has its own Tangent Space. The Tangent Space has the same dimensionality as the Manifold itself.

The Tangent Space at the point  $P$  on Manifold  $M$  is denoted by  $T_P(M)$ . In our diagram we show two Tangent Spaces on the Manifold  $T_{P_2}(M), T_{P_1}(M)$  at the points  $P_2$  and  $P_1$  respectively. A few example vectors are drawn in each Tangent Space. So we require that vectors **only live in tangent spaces**.

In the example of  $S^2$  each tangent space is  $\mathbb{R}^2$ . This makes intuitive sense because

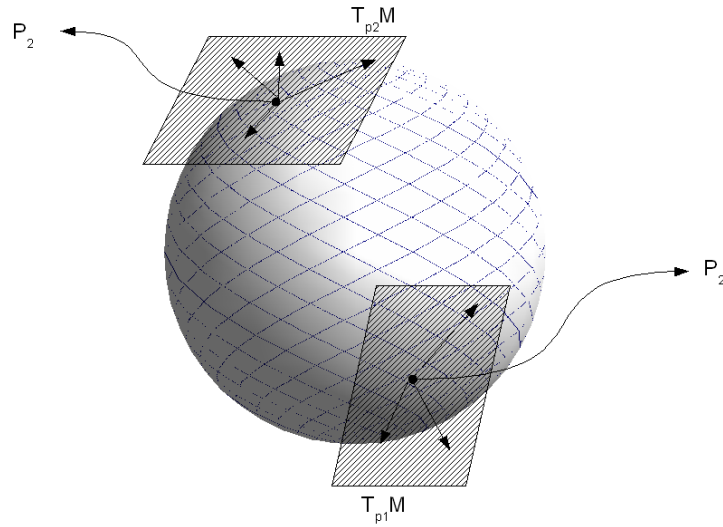
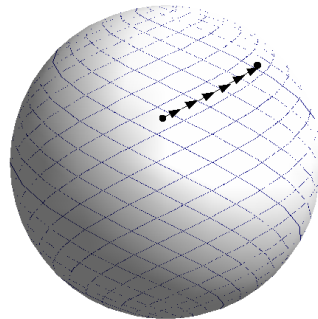


Figure 1.6: Tangent Space

locally,  $S^2$  looks like  $\mathbb{R}^2$ . Similarly for an  $m$  dimensional space, the Tangent Space will be  $\mathbb{R}^m$ . Now how do model displacement from one point to another? We think of displacement as the sum of small vectors (each in their own Tangent Space) adding up (See diagram).



Instead of one single vector in  $\mathbb{R}^n$ , an arbitrary displacement now is the sum of tiny little vectors (each in their own tangent space)

Figure 1.7: How do we get displacement from one point to another?

Let  $T_p(M)$  be the  $m$  dimensional tangent space at some point  $p$  in  $M$ . Let  $\{\hat{e}_1(p), \hat{e}_2(p), \dots, \hat{e}_m(p)\}$  be the basis vectors of the Tangent Space. The reason why we write the basis vectors  $\hat{e}_i(p)$  as a function of the point  $p$  is because the direction of the basis vectors varies in different Tangent Spaces (think of the basis vectors in  $T_{p_2}(M)$  and  $T_{p_1}(M)$ . They are different). We say that the the basis vectors **span** the tangent space. In other words, any vector in a Tangent Space can be expressed as a linear combination of the basis vectors

for that Tangent Space. In the mathematician's notation these basis vectors are written as:

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^m} \right\}$$

There are mathematical reasons why you may want to write basis vectors in the above notation. The concept of partial derivatives captures the concept of basis vectors well. However we do not lose very much if we mentally convert the above expression to  $\{\hat{e}_1(p), \hat{e}_2(p), \dots, \hat{e}_m(p)\}$  for understanding purposes.

### 1.3.2 Vector Fields

A vector field in  $\mathbb{R}^n$  means you have a vector at each point in space. The vector could be in an arbitrary direction. A vector field on a Manifold means that you have a vector at each point on the Manifold. However, the vector direction at a point can only be in *any direction* inside the Tangent Space for that point.



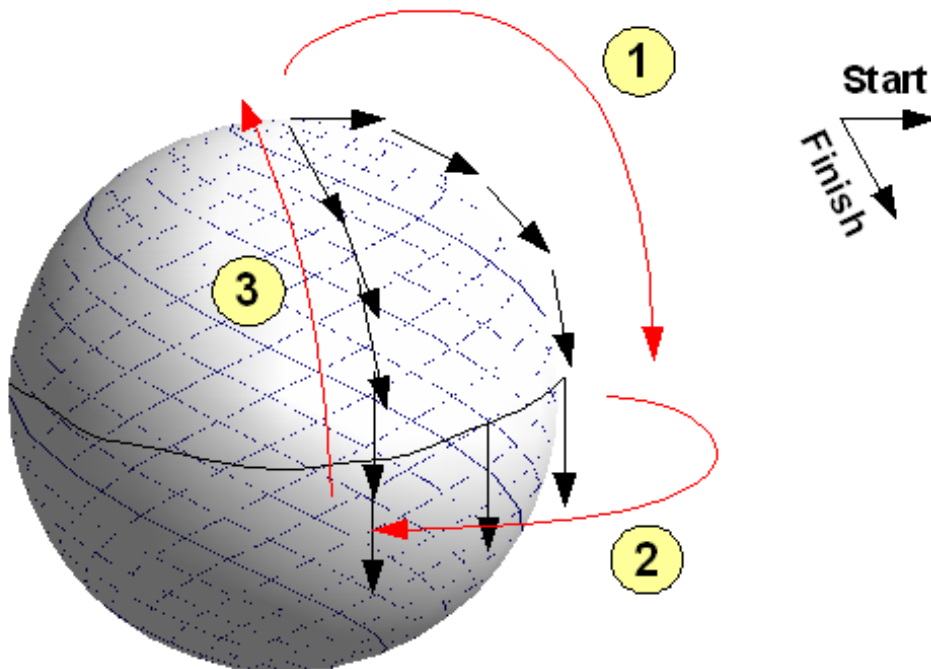
# CHAPTER 2

## RIEMANNIAN MANIFOLDS

### 2.1 Some Preliminaries

To study Riemannian Manifolds and metrics, we first introduce some simple concepts.

#### 2.1.1 Curvature and Parallel Transport



A very intuitive understanding of curvature in space can be obtained by understanding parallel transport on  $S^2$ . Follow the vector along the path shown in the diagram. At each point, the vector is in the Tangent Space of that point. We take care not to shift or twist the vector in any other way. We shift it “parallel to itself.”

A surprising thing happens. Over a closed path, the starting vector is different from the ending vector. When this happens, we say that the Manifold has curvature over the path. This phenomenon does not happen in  $\mathbb{R}^n$ .

We will now measure curvature mathematically. For that we need to understand co-variant derivatives or a connection.

### 2.1.2 Covariant Derivative / Connection

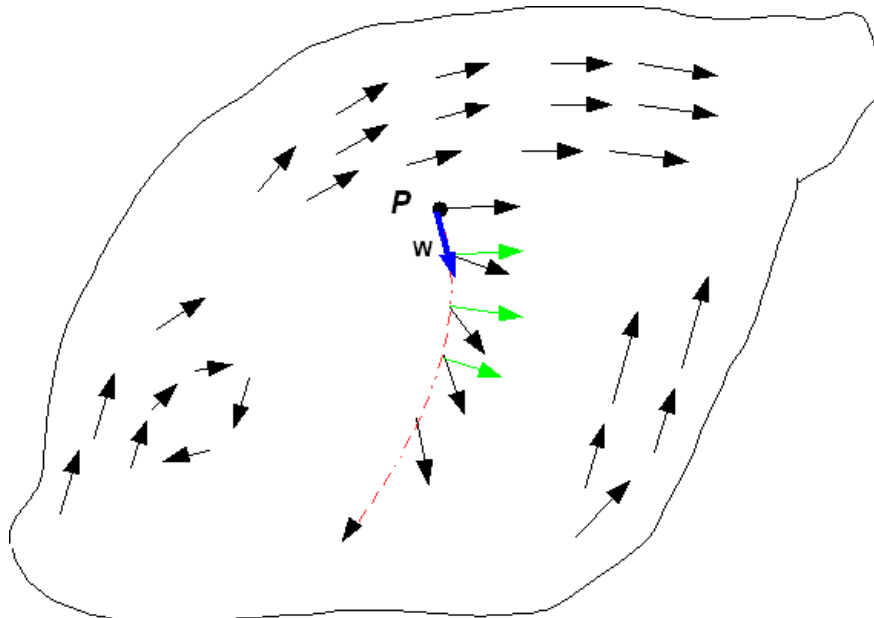


Figure 2.1: Concept of Covariant Derivatives. Diagram adapted from [2, pg. 298]

Suppose we have a smoothly varying vector (or a tensor) field over a Manifold. We should be able to capture the way it is changing by calculating its “derivative.” This derivative should have the same value regardless of the local coordinates we use to calculate it. In other words, we need to define a “covariant” derivative. The covariant derivative allows us express more complicated concepts like curvature and torsion in space.

Look at the Manifold in the above figure. It is a good example of a vector field. We centre our interest on how the vector field varies along the red dashed curve. The vector field at various points on the curve is shown by black arrows. How much is a vector field varying with respect to the vector at point  $P$ ? It is tough to say because the Manifold has curvature (and we simply cannot subtract two vectors at different points on the Manifold to get a difference as we did in  $\mathbb{R}^n$ ). In order to compare the change we parallelly transport the vector from point  $P$ . The parallelly transported vector is depicted in green. Now we can calculate the difference  $\Delta \mathbf{v}$ . In the case of covariant derivatives we are interested in change only in a particular direction and not over a curve. In our diagram the direction that we are interested in is  $\mathbf{w}$ . We think of a covariant derivative of a vector field in the direction  $\mathbf{w}$ .

Assume we are in some local coordinate system with coordinates  $x^\alpha$ . We define the covariant derivative of a vector field  $\mathbf{v}(x^\alpha)$  in direction  $\mathbf{w}$  as:

$$\nabla_{\mathbf{w}} \mathbf{v}(x^\alpha) = \lim_{\epsilon \rightarrow 0} \frac{(\mathbf{v}(x^\alpha + w^\alpha \epsilon))_{\parallel} \text{ transport to } x^\alpha - \mathbf{v}(x^\alpha)}{\epsilon} \text{ (Notation from [3, pg. 433])}$$

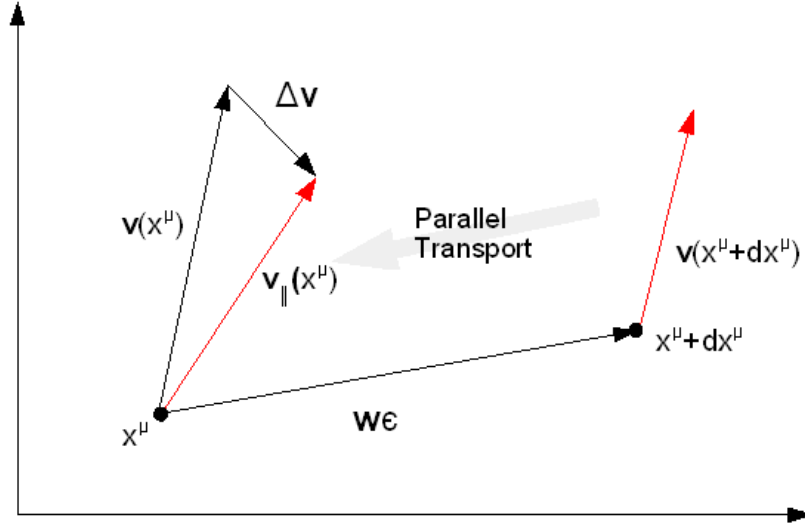


Figure 2.2: Concept of Covariant Derivatives II. Diagram adapted from [3, pg. 433]

Properties of the covariant derivative:

$$\nabla_{\mathbf{w}+\mathbf{u}} = \nabla_{\mathbf{w}} + \nabla_{\mathbf{u}}$$

$$\nabla_{\lambda\mathbf{w}} = \lambda\nabla_{\mathbf{w}}$$

$$\nabla_{\mathbf{w}}(\mathbf{E} + \mathbf{N}) = \nabla_{\mathbf{w}}\mathbf{E} + \nabla_{\mathbf{w}}\mathbf{N}$$

Leibnitz Law:

$$\nabla_{\mathbf{w}}(\lambda\mathbf{E}) = \lambda\nabla_{\mathbf{w}}\mathbf{E} + \mathbf{E}\nabla_{\mathbf{w}}\lambda$$

NOTE:  $\mathbf{E}, \mathbf{N}$  are vector fields.  $\lambda$  is some scalar function.

### 2.1.3 Riemann Curvature

Riemann curvature  $\mathbf{R}(\mathbf{L}, \mathbf{M}, \mathbf{N})$  where  $\mathbf{L}, \mathbf{M}, \mathbf{N}$  are vector fields is defined as:

$$\mathbf{R}(\mathbf{L}, \mathbf{M}, \mathbf{N}) = (\nabla_{\mathbf{L}}\nabla_{\mathbf{M}} - \nabla_{\mathbf{M}}\nabla_{\mathbf{L}} - \nabla_{[\mathbf{L}, \mathbf{M}]})\mathbf{N}$$

$[\mathbf{L}, \mathbf{M}] \equiv \mathbf{L}\mathbf{M} - \mathbf{M}\mathbf{L}$ . Note that  $\mathbf{R}$  maps three vectors to a vector. In tensorial notation:

$$\mathbf{R}(\mathbf{L}, \mathbf{M}, \mathbf{N}) = L^a M^b N^c R_{abc}{}^d$$

$R_{abc}{}^d$  is the Riemannian curvature tensor. Ricci Curvature  $R_{ac}$

$$R_{ac} = R_{adc}{}^d$$

and Ricci scalar  $S$

$$S = g^{ac} R_{ac}$$

## 2.2 Riemannian Manifolds & Metrics

What is  $g^{ac}$ ? It is the metric and is topic of this section.

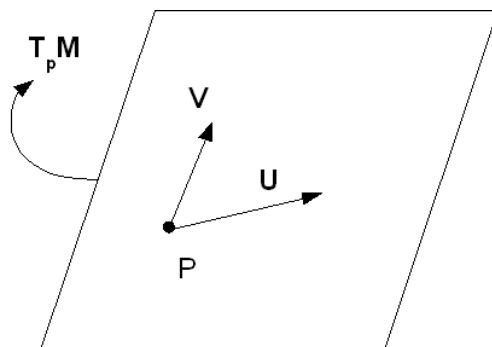


Figure 2.3: What is a metric?

Let  $M$  be a Manifold with a point  $p$  on it. Let  $T_p(M)$  be the Tangent Space at  $p$ . Let  $\mathbf{u}, \mathbf{v}$  be any two vectors in  $T_p(M)$ . What is a metric? A metric at point  $p$  is a map that takes  $\mathbf{u}, \mathbf{v}$  as parameters and gives some number.

$$g_p(\mathbf{u}, \mathbf{v}) = \text{some number}$$

If  $g_p(\mathbf{u}, \mathbf{v}) = g_p(\mathbf{v}, \mathbf{u})$  and  $g_p(\mathbf{u}, \mathbf{u}) \geq 0$  for all points in  $M$  then  $g$  is said to be a Riemannian Metric. A Manifold with a Riemannian Metric is called a Riemannian Manifold.

The notation that we have used is that of a mathematician. However when it comes to metrics in physics we usually use the tensorial notation. We will explain the tensorial notation in terms of the notation above. Think of  $g_{ab} = g_{ab}(p)$  where  $p \in M$ . The  $p$  is implicit as the metric tensor can usually vary from point to point. Furthermore, think of  $g_{ab}$  as  $g_p(\hat{e}_a, \hat{e}_b)$ . In  $\mathbb{R}^3$

$$g_{12} = g_{12}(p) = g_p(\hat{i}, \hat{j})$$

## CHAPTER 3

### COMPLEX MANIFOLDS

#### 3.1 Introduction

In a Complex Manifold, each coordinate is complex. A Complex Manifold of dimension  $m$  looks like  $\mathbb{C}^m$  locally. The transition functions from one local coordinate system to another satisfy the Cauchy-Riemann conditions i.e. are holomorphic.

Let  $z$  be a complex coordinate expressed in terms of its real components.  $z^\mu = x^\mu + iy^\mu$  for some local patch  $U_1$ . Let  $w$  be a complex coordinate expressed in terms of its real components  $w^\nu = u^\nu + iv^\nu$  for another patch  $U_2$ . Let  $U_1 \cap U_2 \neq \emptyset$ . Then the transition functions satisfy the Cauchy-Riemann conditions:

$$\frac{\partial u^\nu}{\partial x^\mu} = \frac{\partial v^\nu}{\partial y^\mu} \quad (3.1)$$

$$\frac{\partial u^\nu}{\partial y^\mu} = -\frac{\partial v^\nu}{\partial x^\mu} \quad (3.2)$$

An equivalent way of saying this is that:

$$w^\nu(z^1, z^2, \dots, z^m, \bar{z}^1, \bar{z}^2, \dots, \bar{z}^m) = w^\nu(z^1, z^2, \dots, z^m)$$

In other words,  $w^\nu$  is not a function of  $\bar{z}^i$ . As always  $1 \leq \mu, \nu \leq m$ .

*Proof:*

If  $w^\nu$  is not a function of  $\bar{z}^i \Rightarrow \frac{\partial w^\nu}{\partial \bar{z}^\mu} = 0$

Now:

$$\begin{aligned} \frac{\partial w^\nu}{\partial \bar{z}^\mu} &= \frac{\partial u^\nu + iv^\nu}{\partial(x^\mu - iy^\mu)} \\ &= \frac{\partial u^\nu}{\partial x^\mu} - \frac{1}{i} \frac{\partial u^\nu}{\partial y^\mu} + i \left( \frac{\partial v^\nu}{\partial x^\mu} - \frac{1}{i} \frac{\partial v^\nu}{\partial y^\mu} \right) \\ &= \left( \frac{\partial u^\nu}{\partial x^\mu} - \frac{\partial v^\nu}{\partial y^\mu} \right) + i \left( \frac{\partial v^\nu}{\partial x^\mu} + \frac{\partial u^\nu}{\partial y^\mu} \right) \\ &= 0 + i0 = 0 \text{ (from 3.1 and 3.2)} \end{aligned}$$

Hence proved.

### 3.2 The Stereographic Projection Example: Continued

Stereographic coordinates of a point  $P(x, y, z)$  on a unit sphere

North Pole Projection:

$$(X, Y) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right)$$

South Pole Projection:

$$(U, V) = \left( \frac{x}{1+z}, \frac{-y}{1+z} \right)$$

Let

$$Z = X + iY \Rightarrow \bar{Z} = X - iY$$

Let

$$\begin{aligned} W &= U + iV \Rightarrow \bar{W} = U - iV \\ W &= \frac{x - iy}{1+z} \\ &= \frac{1-z}{1+z} \left( \frac{x}{1-z} - \frac{iy}{1-z} \right) \\ &= \frac{1-z}{1+z} (X - iY) \\ &= \frac{X - iY}{\frac{1+z}{1-z}} = \frac{X - iY}{X^2 + Y^2} \\ &= \frac{1}{\bar{Z}} \end{aligned}$$

$$\boxed{W(Z, \bar{Z}) = \frac{1}{\bar{Z}}}$$

$W$  does not depend on  $\bar{Z}$  Therefore  $S^2$  is a complex manifold of complex dimension 1

### 3.3 Some Formalism

Tangent Space  $T_p(M)$  of a Real Manifold of dimension  $m$  is spanned by  $m$  vectors

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^m} \right\}$$

Tangent Space  $T_p(M)$  of a Complex Manifold of dimension  $m$  is spanned by  $2m$  vectors

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^m}; \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \dots, \frac{\partial}{\partial y^m} \right\}$$

Note that we are in some arbitrary local patch with coordinates  $z^\mu = x^\mu + iy^\mu$ . We define:

$$\frac{\partial}{\partial z^\mu} \equiv \frac{1}{2} \left\{ \frac{\partial}{\partial x^\mu} - i \frac{\partial}{\partial y^\mu} \right\}$$

$$\frac{\partial}{\partial \bar{z}^\mu} \equiv \frac{1}{2} \left\{ \frac{\partial}{\partial x^\mu} + i \frac{\partial}{\partial y^\mu} \right\}$$

Given the above definitions we can alternatively say that the tangent space  $T_p(M)$  of the Complex Manifold is spanned by:

$$\left\{ \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \dots, \frac{\partial}{\partial z^m}; \frac{\partial}{\partial \bar{z}^1}, \frac{\partial}{\partial \bar{z}^2}, \dots, \frac{\partial}{\partial \bar{z}^m} \right\}$$

Let us define a map  $J_p$  such that:

$$J_p \partial / \partial x^\mu = \partial / \partial x^\mu \text{ and } J_p \partial / \partial y^\mu = -\partial / \partial x^\mu$$

from the Cauchy-Riemann relations we can show that

$$J_p \partial / \partial u^\mu = \partial / \partial v^\mu \text{ and } J_p \partial / \partial v^\mu = -\partial / \partial u^\mu$$

where  $z^\mu = x^\mu + iy^\mu$  and  $w^\mu = u^\mu + iv^\mu$  are local coordinates on two patches. What is interesting is that the Cauchy Riemann conditions on the transition functions allows  $J_p$  to have the same consistent definition on another patch. So while we can have  $J_p$  on any patch of a manifold it is only in complex manifolds that  $J_p$  “spreads” like this.

### 3.4 Hermetian Manifolds

Hermetian Manifolds are Manifolds which satisfy

$$g_p(J_p X, J_p Y) = g_p(X, Y)$$

### 3.5 Kähler Manifolds

Kähler Manifolds are a special kind of Hermetian Manifold in which the metric  $g$  can be expressed in terms of a potential function  $K$

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K$$

This is useful because we need to only keep track of the potential instead of a whole metric tensor.

# CHAPTER 4

## ORBIFOLDS

Orbifolds are a special kind of Manifold. To understand Orbifolds we first need to understand *equivalence relations*. An equivalence relation  $\sim$  is a relation which satisfies:

- $a \sim a$
- if  $a \sim b$  then  $b \sim a$
- if  $a \sim b$  and  $b \sim c$  then  $a \sim c$

According to [1, pg. 38] “Given a set  $X$  and an equivalence relation  $\sim$ , we have a partition of  $X$  into *mutually disjoint* subsets called **equivalence classes**. A class  $[a]$  is made of all the elements  $x$  in  $X$  such that  $x \sim a$ .  $[a] \equiv \{x \in X | x \sim a\}$ ”

The set of all the mutually disjoint classes referred to above is called **quotient space**. The notation is  $X/\sim$ . We motivate a very elegant example (from [1, pg. 39]). Let  $x, y \in \mathbb{R}$ . Also let  $x \sim y$  iff  $y = x + 2\pi n$ ,  $n \in \mathbb{Z}$ . Now  $[x] = \{\dots, x - 2\pi n, x + 2\pi n, x + 4\pi n, \dots\}$ . A little bit of thought convinces us that  $\mathbb{R}/\sim = S^1$  where  $S^1$  is a circle. We look at another example (from [1, pg. 41]). Let us have a Disc  $D^2$  of radius 1 on  $\mathbb{R}^2$ .

$$D^2 = \{x^2 + y^2 \leq 1, x, y \in \mathbb{R}\}$$

. Two points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $D^2$  are equivalent iff  $x_1^2 + y_1^2 = 1$  and  $x_2^2 + y_2^2 = 1$  (both are on the edge of the Disc). All points on the edge of the disc are equivalent. Now imagine a rubber sheet shaped like a disc. If we stuck all the edges of the disc together we would get a sphere  $S^2$ . Therefore  $D^2/\sim = S^2$ . There is another way of saying this. All the points on the edge of the disc were equivalent and they made a circle  $S^1$ . So  $D^2/S^1 = S^2$ .

In general if we have quotient space  $A = M/B$  and if  $M$  is a Manifold and  $B$  is a **discrete** group acting on  $M$  then  $A$  is an Orbifold. A simple example is  $\mathbb{C}/Z_3$ .  $Z_3 = \{1, \omega, \omega^2\}$  (cube roots of unity). Lets take  $z \in \mathbb{C}$ . Now  $\{z, z\omega, z\omega^2\}$  are all equivalent to each other. We are left with only one third of the complex plane. This one third is the Orbifold  $\mathbb{C}/Z_3$ .



# CHAPTER 5

## NUMERICAL COMPUTATIONS

For this part of the project three tasks were performed:

- The  $\mathbb{C}^3/\mathbb{Z}_3$  orbifold was represented in terms of a polytope and visualized in 3D
- The  $\mathbb{C}^3/\mathbb{Z}_5$  orbifold was represented in terms of a polytope and visualized in 3D
- Given a potential for  $\mathbb{C}^3/\mathbb{Z}_3$  various metrics (Ricci flat and non flat) were calculated. After calculating the metric, the Ricci tensor was calculated. The Ricci scalar was also calculated and represented graphically.

### 5.1 Visualization of $\mathbb{C}^3/\mathbb{Z}_3$

The  $\mathbb{C}^3/\mathbb{Z}_3$  orbifold can be represented mathematically as a polytope <sup>1</sup>. with the following properties:

- $x_i > 0$  where  $i = 1, 2, 3$
- $(t - x_1 - x_2 + 3x_3) > 0$  where  $t =$  blow-up parameter (a constant)

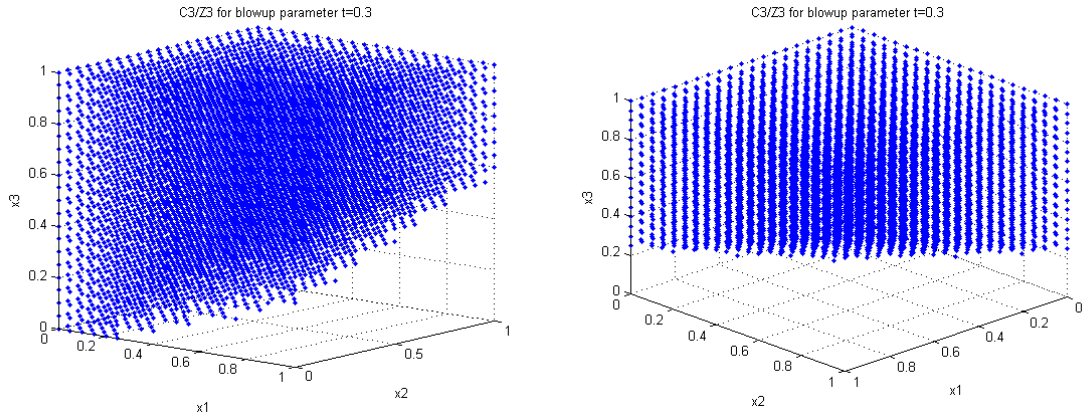


Figure 5.1: The  $\mathbb{C}^3/\mathbb{Z}_3$  polytope from two different angles.  $t = 0.3$

The program to achieve this is placed in Appendix A.

<sup>1</sup>According to the Wikipedia “In geometry polytope means, first, the generalization to *any* dimension of polygon in two dimensions, polyhedron in three dimensions, and polychoron in four dimensions”

## 5.2 Visualization of $\mathbb{C}^3/\mathbb{Z}_5$

$\mathbb{C}^3/\mathbb{Z}_5$  orbifold can be represented mathematically as a polytope with the following properties:

- $x_i > 0$  where  $i = 1, 2, 3$
- $t_1 - x_1 - 3x_2 - x_3 > 0$  and
- $t_2 - x_2 - 2x_3 > 0$  where  $t_1, t_2 =$  blow-up parameters (two constants)

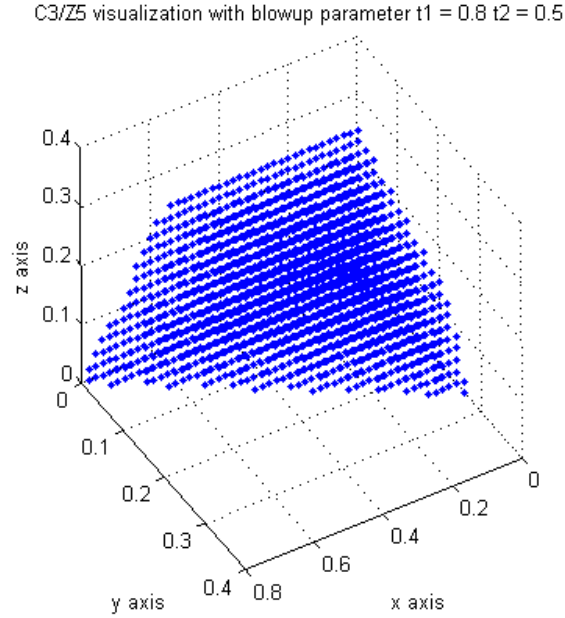


Figure 5.2: c3modz5

The program to achieve this has been placed in Appendix B.

## 5.3 Calculation of Ricci Scalar for $\mathbb{C}^3/\mathbb{Z}_3$

- $y_i > 0$  where  $i = 1, 2, 3$
- $(t - y_1 - y_2 + 3y_3) > 0$  where  $t =$  blow-up parameter (a constant)

For  $\mathbb{C}^3/\mathbb{Z}_3$ , given a potential  $G(y)$  we can calculate the metric:

$$g_{ij} = \frac{\partial^2 G(y)}{\partial y_i \partial y_j}$$

$g^{ij} =$  inverse of the matrix  $g_{ij}$

Given the metric we can calculate the Ricci Tensor and Scalar:

$$R_{cd} = \text{Ricci Tensor} = -\frac{1}{2}g^{fd} \frac{\partial^2 g^{ce}}{\partial y_e \partial y_f}$$

$$R = \text{Ricci Scalar} = R_{ij}g_{ij}$$

See [4] for a detailed treatment of above formulae. Now, given a seed potential:

$$G_p = \frac{1}{2}(y_1 \ln y_1 + y_2 \ln y_2 + y_3 \ln y_3 + (t - y_1 - y_2 + 3y_3) \ln (t - y_1 - y_2 + 3y_3))$$

And a correction term  $f(y)$  such that:

$$f(y) = \frac{1}{2}((y - \lambda t) \ln(y - \lambda t) + (y - \lambda^* t) \ln(y - \lambda^* t) - (3y + t) \ln(3y + t))$$

A Ricci flat metric can be obtained from the potential  $G = G_p + f(y_3)$

We perform the following steps on the computer:

- We create a representation of the  $\mathbb{C}^3/\mathbb{Z}_3$  polytope with a discrete set of points in a volume from say  $(0.04, 0.04, 0.04)$  to  $(1, 1, 1)$ . Each point may be separated from another by a small enough  $\delta$
- We calculate the Ricci Scalar at each point in the polytope given seed potential  $G_p$
- We calculate the Ricci Scalar at each point in the polytope given potential  $G = G_p + f(y_3)$
- We take a slice of the polytope at some  $y_3 = 0.7$  (say) and depict the absolute value of the Ricci scalar on the  $z$ -axis and the  $y_1$  and  $y_2$  coordinates on the  $x$  and  $y$  axes respectively

The programs to achieve this have been placed in Appendix C,D. A small note on using the programs is given in Appendix E.

#### 5.4 Looking at the absolute value of the Ricci Scalars

We take a slice of the polytope at some  $y_3 = 0.7$  (say) and depict the absolute value of the Ricci scalar on the  $z$ -axis and the  $y_1$  and  $y_2$  coordinates on the  $x$  and  $y$  axes respectively. We perform this for the seed metric  $G_p$  and the Ricci-flat metric  $G_p + f(y_3)$ . We get the following figures (for minimum separation between points as  $0.02$  ( $\delta$ )):

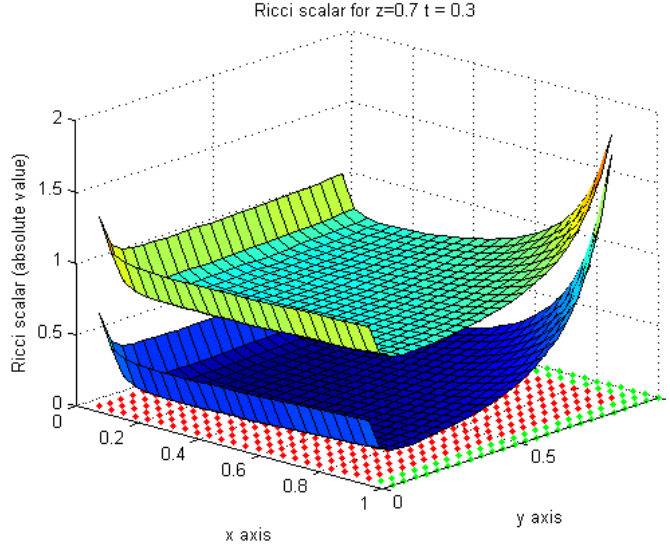


Figure 5.3: Looking at Ricci scalars produced by potential  $G = G_p$  (above) and  $G = G_p + f(y_3)$  (below)

The red points in the graph are the  $(y_1, y_2)$  points where the Ricci Scalar exists (and the polytope exists). The green points in the graph are the points where the polytope exists but we do not know the value of the Ricci Scalar as we don't have enough values for numerical differentiation.

Notice, that except at the edges, the surface of the curve is flat. In fact, this edge behaviour is due to a numerical artifact of computation. The theoretical value of the Ricci Scalar for the Guillemin metric (see [4, pg. 6] is:

$$\frac{12(t^2 + 72y_3^2)}{(t + 12y_3)^3}$$

For  $t = 0.3$  and  $y_3 = 0.7$  (we also call it  $z$  in the graph) theoretical value of  $S = 0.64$  (to two decimal places). We get experimental value of 0.82 (when  $\delta = 0.02$  and the boundaries of the polytope are from  $(0.04, 0.04, 0.04)$  to  $(1, 1, 1)$ )

The next graph is same as the previous graph except we now show the value of the potential in cyan colour.

**Does the Ricci Scalar reach its theoretical value of 0.64 for  $G = G_p$ ?**

We cannot numerically prove this but as we decrease the  $\delta$ , our values should converge

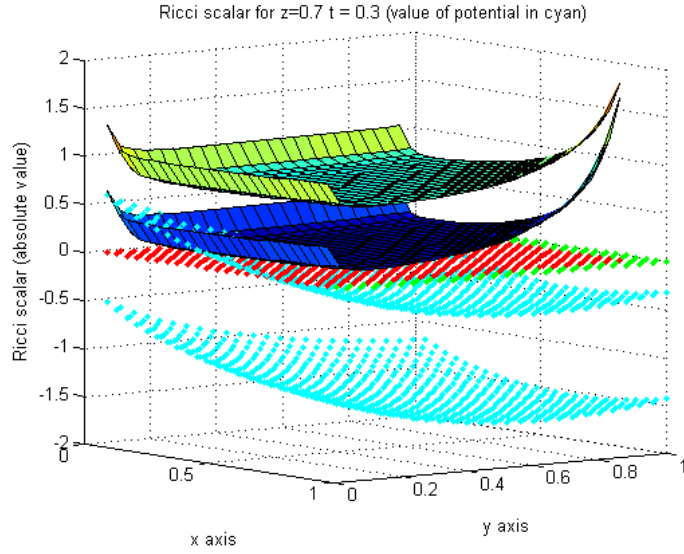


Figure 5.4: Looking at potential (in cyan)  $G = G_p$  (above) and  $G = G_p + f(y_3)$  (below) to 0.64. For polytope boundary  $(0.04, 0.04, 0.04)$  to  $(1, 1, 1)$

$\delta$	Ricci Scalar Average
0.05	1.05
0.04	1.00
0.03	0.93
0.02	0.82
0.015	0.77

### Does the Ricci Scalar Vanish for $G = G_p + f(y_3)$ ?

Again, we cannot numerically prove this but we can however establish a trend. As we decrease *delta* we should see the Ricci Scalar Average tend to 0. For polytope boundary  $(0.04, 0.04, 0.04)$  to  $(1, 1, 1)$

$\delta$	Ricci Scalar Average
0.05	0.43
0.04	0.37
0.03	0.29
0.02	0.18
0.015	0.12

## CONCLUSION

We have given an comprehensive introduction to Manifolds. We looked at the whole hierarchy of Manifolds. We started from Simple Manifolds and progressed to Differentiable Manifolds, Riemannian Manifolds and Complex Manifolds. Within Complex Manifolds we studied Hermetian Manifolds and Kähler Manifolds. Orbifolds, another special kind of Manifold, were also introduced. All the related mathematics and concepts such as Vector Fields, Tangent Spaces, Metrics, Curvature, Parallel Transport and Connection were explained.

In Chapter 5 we also performed some numerical computations on  $\mathbb{C}^3/\mathbb{Z}_3$  and  $\mathbb{C}^3/\mathbb{Z}_5$ .

We have gained theoretical and computational knowledge about Manifolds and Differential Geometry in this project.

## APPENDIX A

Code for c3modz3\_polytope\_visualize.m

```
% Visualize the C3/Z3 Polytope

% t is the blowup parameter
t=.3;
clear x1p;
clear x2p;
clear x3p;

% Generate points in the polytope
j=0;
delta=0.05;
for x1=0:delta:1
    for x2=0:delta:1
        for x3=0:delta:1
            if t-x1-x2+3*x3 > 0
                j=j+1;
                x1p(j) = x1;
                x2p(j) = x2;
                x3p(j) = x3;
            end
        end
    end
end

grid on;
hold on;
xlabel('x1');
ylabel('x2');
zlabel('x3');
title('C3/Z3 for blowup parameter t=0.3');

plot3(x1p,x2p,x3p,'.');
```

## APPENDIX B

Code for c3modz5\_polytope\_visualize.m

```
% Visualize the C3/Z5 polytope
j=0;
% t1, t2 are the blowup parameters
t1=.8;t2=.5;
clear x1p;
clear x2p;
clear x3p;

delta = 0.02;
endx=2;endy=2;endz=2;
for x1=0.0:delta:endx
    for x2=0.0:delta:endy
        for x3=0.0:delta:endz
            if t1-x1-3*x2-x3 > 0 && t2-x2-2*x3 > 0
                j=j+1;
                x1p(j) = x1;
                x2p(j) = x2;
                x3p(j) = x3;
            end
        end
    end
end

x = [x1p; x2p; x3p];
hold on;
grid on;
axis square;
title(['C3/Z5 visualization with blowup parameter t1 = ',num2str(t1),
' t2 = ',num2str(t2)]);
xlabel('x axis');
ylabel('y axis');
zlabel('z axis');
plot3(x1p,x2p,x3p,'.');
hold off;
```



## APPENDIX C

Code for c3modz3\_metrics\_and\_ricci.m (Please excuse the bad formatting. The code lines were long and have been made to wrap around so that they fit on the page)

```
function [polytope,G ,g_high ,g_low , Ricci] = c3modz3_metrics_and_ricci
(startx, starty, startz, endx, endy, endz, t, delta,guil)
%Example parameters
%t=.3;startx=0.04;starty=0.04;startz=0.04;endx=1;endy=1;endz=1;delta=0.04;
%guil=false

% startx, starty, startz should be small and positive otherwise we will
% have log of zero issues in the calculation of the guillemin metric
% if guil == true then only the guillemin metric is used to calculate the
% Ricci tensors

% g_low = g_{ij} = \frac{\partial^2 G(x)}{\partial x^i \partial x^j}
% G(x) is the symplectic potential (see report for expression)
% g_high = g^{ij} = matrix inverse of g_{ij}
% Ricci = R_{cd} = Ricci Tensor = -0.5 g^{fd} \frac{\partial^2 g^{ce}}{\partial x^e \partial x^f}
%
% R = Ricci Scalar = g_{ij} R_{ij}
% See Arxiv hep-th/9803192

% Generate points in the polytope
j=0;k=0;l=0;
lengx = length(startx:delta:endx);
lengy = length(starty:delta:endy);
lengz = length(startz:delta:endz);

polytope.startx = startx;
polytope.starty = starty;
polytope.startz = startz;
polytope.endx = endx;
polytope.endy = endy;
polytope.endz = endz;
polytope.t = t;
polytope.delta = delta;
```

```

polytope.guil = guil;

polytope.lengx = lengx;
polytope.lengy = lengy;
polytope.lengz = lengz;

polytope.exists = zeros(lengx, lengy, lengz, 'uint8');
for x1=startx:delta:endx
    j=j+1;
    for x2=starty:delta:endy
        k=k+1;
        for x3=startz:delta:endz
            l=l+1;
            if t-x1-x2+3*x3 > 0
                polytope.exists(j,k,l) = true;
            else
                polytope.exists(j,k,l) = false;
            end
        end
        l=0;
    end
    k=0;
end
j=0;

lamb = (-3+i*sqrt(3))/6;
G=zeros(lengx, lengy, lengz);
for k=1:lengx
    for l=1:lengy
        for m=1:lengz
            x = startx+(k-1)*delta;
            y = starty+(l-1)*delta;
            z = startz+(m-1)*delta;

            if(polytope.exists(k,l,m))
                Gp = 1/2 * (x*log(x) + y*log(y) + z*log(z)) + 1/2 *
(t-x-y+3*z)*log(t-x-y+3*z);
            end
        end
    end
end

```

```

        if (guil)
            G(k,l,m) = Gp;
        else
            f = 1/2*((z-lamb*t)*log(z-lamb*t) + (z-conj(lamb)*t)
*log(z-conj(lamb)*t) - (3*z+t)*log(3*z+ t));
            G(k,l,m) = Gp + f;
        end
    else
        G(k,l,m) = NaN;
    end
end
end
end

% calculate g tensor at each point (k,l,m)
nan_array = [NaN NaN NaN;NaN NaN NaN;NaN NaN NaN];
A = zeros(2,2);
for k=1:lengx-2
    for l=1:lengy-2
        for m=1:lengz-2
            A(1,1) = ((G(k+2,l,m)-G(k+1,l,m)) - (G(k+1,l,m) - G(k,l,m)))
/(delta^2);
            A(2,2) = ((G(k,l+2,m)-G(k,l+1,m)) - (G(k,l+1,m) - G(k,l,m)))
/(delta^2);
            A(3,3) = ((G(k,l,m+2)-G(k,l,m+1)) - (G(k,l,m+1) - G(k,l,m)))
/(delta^2);

            % Actually we could use symmetry properties in which A(2,1) =
            % A(1,2) and so on but we avoid that for now so that the least
            % assumptions are made as possible.
            A(2,1) = ((G(k+1,l+1,m)-G(k,l+1,m)) - (G(k+1,l,m) - G(k,l,m)))
/(delta^2);
            A(1,2) = ((G(k+1,l+1,m)-G(k+1,l,m)) - (G(k,l+1,m) - G(k,l,m)))
/(delta^2);

            A(3,1) = ((G(k+1,l,m+1)-G(k,l,m+1)) - (G(k+1,l,m) - G(k,l,m)))
/(delta^2);

```

```

        A(1,3) = ((G(k+1,l,m+1)-G(k+1,l,m)) - (G(k,l,m+1) - G(k,l,m)))
/(delta^2);

        A(3,2) = ((G(k,l+1,m+1)-G(k,l,m+1)) - (G(k,l+1,m) - G(k,l,m)))
/(delta^2);
        A(2,3) = ((G(k,l+1,m+1)-G(k,l+1,m)) - (G(k,l,m+1) - G(k,l,m)))
/(delta^2);

%Are there any NaNs?
if(sum(sum(isnan(A))) > 0)
    g_high(k,l,m).tensor = nan_array;
    g_high(k,l,m).exists = false;
    g_low(k,l,m).tensor = nan_array;
    g_low(k,l,m).exists = false;

else
    B = inv(A);
    g_low(k,l,m).tensor = A;
    g_low(k,l,m).exists = true;
    g_high(k,l,m).tensor = B;
    g_high(k,l,m).exists = true;
end
end
end
end

%Calculate Ricci tensor for each point
R=zeros(3,3);
for k=1:lengx-4
    for l=1:lengy-4
        for m=1:lengz-4
            ricci_exists = true;
            % Calculate Ricci tensor
            for c=1:3
                for d=1:3
                    % e = 1, f = 1
                    dd(1,1) = -1/2*g_high(k,l,m).tensor(1,d)*

```

```

((g_high(k+2,l,m).tensor(c,1) - g_high(k+1,l,m).tensor(c,1)) -
(g_high(k+1,l,m).tensor(c,1) - g_high(k,l,m).tensor(c,1)))/(delta^2);
    % e = 2, f = 2
    dd(2,2) = -1/2*g_high(k,l,m).tensor(2,d)*
((g_high(k,l+2,m).tensor(c,2) - g_high(k,l+1,m).tensor(c,2)) -
(g_high(k,l+1,m).tensor(c,2) - g_high(k,l,m).tensor(c,2)))/(delta^2);
    % e = 3, f = 3
    dd(3,3) = -1/2*g_high(k,l,m).tensor(3,d)*
((g_high(k,l,m+2).tensor(c,3) - g_high(k,l,m+1).tensor(c,3)) -
(g_high(k,l,m+1).tensor(c,3) - g_high(k,l,m).tensor(c,3)))/(delta^2);

    % Actually we could use symmetry properties in which d(2,1) =
    % d(1,2) and so on but we avoid that for now so that the least
    % assumptions are made as possible.
    % e = 2, f = 1
    dd(2,1) = -1/2*g_high(k,l,m).tensor(1,d)*
((g_high(k+1,l+1,m).tensor(c,2) - g_high(k,l+1,m).tensor(c,2)) -
(g_high(k+1,l,m).tensor(c,2) - g_high(k,l,m).tensor(c,2)))/(delta^2);
    % e = 1, f = 2
    dd(1,2) = -1/2*g_high(k,l,m).tensor(2,d)*
((g_high(k+1,l+1,m).tensor(c,1) - g_high(k+1,l,m).tensor(c,1)) -
(g_high(k,l+1,m).tensor(c,1) - g_high(k,l,m).tensor(c,1)))/(delta^2);

    % e = 3, f = 1
    dd(3,1) = -1/2*g_high(k,l,m).tensor(1,d)*
((g_high(k+1,l,m+1).tensor(c,3) - g_high(k,l,m+1).tensor(c,3)) -
(g_high(k+1,l,m).tensor(c,3) - g_high(k,l,m).tensor(c,3)))/(delta^2);
    % e = 1, f = 3
    dd(1,3) = -1/2*g_high(k,l,m).tensor(3,d)*
((g_high(k+1,l,m+1).tensor(c,1) - g_high(k+1,l,m).tensor(c,1)) -
(g_high(k,l,m+1).tensor(c,1) - g_high(k,l,m).tensor(c,1)))/(delta^2);

    % e = 3, f = 2
    dd(3,2) = -1/2*g_high(k,l,m).tensor(2,d)*
((g_high(k,l+1,m+1).tensor(c,3) - g_high(k,l,m+1).tensor(c,3)) -
(g_high(k,l+1,m).tensor(c,3) - g_high(k,l,m).tensor(c,3)))/(delta^2);
    % e = 2, f = 3

```

```

        dd(2,3) = -1/2*g_high(k,l,m).tensor(3,d)*
((g_high(k,l+1,m+1).tensor(c,2) - g_high(k,l+1,m).tensor(c,2)) -
(g_high(k,l,m+1).tensor(c,2) - g_high(k,l,m).tensor(c,2)))/(delta^2);

        %Are there any NaNs?
        R(c,d) = sum(sum(dd));
        if(R(c,d) == NaN)
            %The ricci tensor does not exist for k,l,m
            %combination
            ricci_exists = false;
            R = nan_array;
            break;
        end
    end %d
    if(ricci_exists == false)
        break;
    end
end %c

Ricci(k,l,m).exists = ricci_exists;
Ricci(k,l,m).tensor = R;
% Calculate Ricci Scalar
ricci_scalar = 0;
if ricci_exists
    for p=1:3
        for q=1:3
            ricci_scalar = ricci_scalar +
g_low(k,l,m).tensor(p,q)*Ricci(k,l,m).tensor(p,q);
        end
    end
end
Ricci(k,l,m).scalar = ricci_scalar;
end %m
end %l
end %k

```

## APPENDIX D

Code for visualize\_c3modz3\_ricci.m

```
function visualize_c3modz3_ricci(polytope, G, Ricci, visualize_G, z, skip)
% Visualize the curvature (Ricci Scalar) through the polytope
% Can visualize the polytope at various slices of z
% Ricci scalar value is on the z-axis
% if visualize_G == true then the potential is graphed
% Sometimes there are too many data points so we may want to skip, say,
% every 4 or 8. so set skip parameter to 8. default should be 1

hold on;
grid on;

startx = polytope.startx;
starty = polytope.starty;
startz = polytope.startz;

endx = polytope.endx;
endy = polytope.endy;
endz = polytope.endz;

delta = polytope.delta;
t = polytope.t;

lengx = polytope.lengx;
lengy = polytope.lengy;
lengz = polytope.lengz;

m=floor((z-startz)/delta);
for k=1:skip:lengx-4
    for l=1:skip:lengy-4
        height(k,l)=abs(Ricci(k,l,m).scalar);
    end
end

% Mark x,y points at given z for which the polytope exists
```

```

for k=1:skip:lengx
    for l=1:skip:lengy
        x = startx + (k-1)*delta;
        y = starty + (l-1)*delta;

        if(polytope.exists(k,l,m))
            plot3(x,y,0,'.g');
        end

        if(visualize_G)
            plot3(x,y,G(k,l,m),'.c');
        end
    end
end

% Mark x,y points at given z for which the Ricci tensor exists
ricci_scalar_sum = 0;
for k=1:skip:lengx-4
    for l=1:skip:lengy-4
        x = startx + (k-1)*delta;
        y = starty + (l-1)*delta;

        if(Ricci(k,l,m).exists)
            plot3(x,y,0,'.r');
        end
    end
end

% Calculate ricci_scalar_sum
% Do it without the skips
for k=1:lengx-4
    for l=1:lengy-4
        x = startx + (k-1)*delta;
        y = starty + (l-1)*delta;

        if(Ricci(k,l,m).exists)
            ricci_scalar_sum = ricci_scalar_sum + Ricci(k,l,m).scalar;
        end
    end
end

```



```

        end
    end
end

ricci_scalar_avg = ricci_scalar_sum / ((lengy-4)*(lengx-4))
disp('Theoretical value of seed metric ricci scalar');
seed_metric_ricci_scalar = 12*(t^2+72*z^2)/(t+12*z)^3

xlabel('x axis');
ylabel('y axis');

zlabel('Ricci scalar (absolute value)');
if(visualize_G)
    title(['Ricci scalar for z=',num2str(z),' t = ', num2str(t),\ ' (value
        of potential in cyan)']);
else
    title(['Ricci scalar for z=',num2str(z),' t = ', num2str(t)]);
end

x = startx:delta:endx;
y = starty:delta:endy;

[xg, yg] = meshgrid(x(1:skip:lengx-4),y(1:skip:lengy-4));
surf(xg, yg, height(1:skip:lengx-4, 1:skip:lengy-4));

```

# APPENDIX E

## Sample Command Invocations

In order to generate a  $\mathbb{C}^3/\mathbb{Z}_3$  polytope from (0.04,0.04,0.04) to (1,1,1) with a distance between adjacent points 0.04 we issue the following command at the MATLAB command line:

```
[poly,G,g_l,g_h,ricci]=c3modz3_metrics_and_ricci(0.04,0.04,0.04,1,1,1,0.3,0.04,false);
```

Please see the program code for understanding details about various options.

- **poly** structure contains information on which points the polytope is present in, between (0.04,0.04,0.04) to (1,1,1). It also contains some other book keeping information about the  $\mathbb{C}^3/\mathbb{Z}_3$  polytope.
- **G** contains the value of the potential at each point in the polytope.
- **g\_l** structure contains the value of  $g_{ij}$  at each point in the polytope.
- **g\_h** structure contains the value of  $g^{ij}$  at each point in the polytope.
- **ricci** structure contains the Ricci tensor and the Ricci scalar for each point on the polytope.

To visualize the Ricci Scalar for the polytope at the  $z=0.7$  slice we issue the command:

```
visualize_c3modz3_ricci(poly,G,ricci,false,0.7,1)
```

We may want to compare the Ricci scalars with the potentials  $G = G_p$  and  $G = G_p + f(y_3)$ . In such cases we issue the following 4 commands in sequence:

```
[polyt,Gt,glt,ght,riccit]=c3modz3_metrics_and_ricci(0.04,0.04,0.04,1,1,1,0.3,0.04,true);  
[poly,G,gl,gh,ricci]=c3modz3_metrics_and_ricci(0.04,0.04,0.04,1,1,1,0.3,0.04,false);  
visualize_c3modz3_ricci(polyt,Gt,riccit,false,0.7,1)  
visualize_c3modz3_ricci(poly,G,ricci,false,0.7,1)
```

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## SOME OTHER USEFUL RESOURCES

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